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LINEAR TIME-VARIANT SPACE-VARIANT FILTERS
AND THE WKB APPROXIMATION

by

Lawrence J. Ziomek

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NAVAL POSTGRADUATE SCHOOL
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Wave propagation in a random, inhomogeneous ocean is treated as transmission thru a linear, time-variant, space-variant, random communication channel. A consistent notation (via-a-vis ad hoc), fundamental input-output relations, and various time-space transformations for both deterministic and random linear, time-variant, space-variant, filters are established. Using the method of separation of variables and the W.K.B. approximation, a time-invariant, space-variant, random transfer function of the ocean volume is derived. The ocean volume is characterized by a random index of refraction

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which is a function of depth. The index of refraction is decomposed into a deterministic component and a zero mean random component. In addition, two example calculations are made. The first example involves the derivation of the equations for the random, output electrical signals at each element in a receive planar array of complex weighted point sources in terms of the frequency spectrum of the transmitted electrical signal, the transmit and receive arrays, and the transfer function of the ocean medium. The second example involves the derivation of the coherence function, i.e., the autocorrelation function of the transfer function.

ABSTRACT

Wave propagation in a random, inhomogeneous ocean is treated as transmission thru a linear, time-variant, space-variant, random communication channel. A consistent notation (vis-a-vis ad hoc), fundamental input-output relations, and various time-space transformations for both deterministic and random linear, time-variant, space-variant, filters are established. Using the method of separation of variables and the W.K.B. approximation, a time-invariant, space-variant, random transfer function of the ocean volume is derived. The ocean volume is characterized by a random index of refraction which is a function of depth. The index of refraction is decomposed into a deterministic component and a zero mean random component. In addition, two example calculations are made. The first example involves the derivation of the equations for the random, output electrical signals at each element in a receive planar array of complex weighted point sources in terms of the frequency spectrum of the transmitted electrical signal, the transmit and receive arrays, and the transfer function of the ocean medium. The second example involves the derivation of the coherence function, i.e., the auto-correlation function of the transfer function.

I. INTRODUCTION

Since the wave equation for small amplitude acoustic signals is linear, we can represent the ocean medium as a linear, time-variant, space-variant, random filter (system or communication channel) in general. With this interpretation in mind, refer to Fig. 1 which illustrates a basic bistatic communication channel geometry, and Fig. 2, which is a mathematical block diagram representation of Fig. 1. With respect to Fig. 1, both the transmit and receive apertures (arrays) are, in general, volume apertures (arrays) and in motion. Before proceeding further, a word of caution concerning Fig. 2. Note, for example, that $X_M \neq XD_T$, $Y_M \neq X_M H_M$, and $Y \neq Y_M D_R$ in general. The equations required for describing the filter's input-output relationships and for coupling the transmitted and received electrical signals to the medium via the transmit and receive apertures are developed in Sections II and III, respectively.

Let us now describe the notation used in Fig. 2. The position vectors $\underline{r}_o$ and $\underline{r}$ refer to spatial coordinates (x_o, y_o, z_o) and (x,y,z), respectively, and t refers to time in sec. The parameters f and η are frequencies in HZ. where f represents input or transmitted frequencies while η represents output or received frequencies. Note, that if $\eta \neq f$, Doppler spread is implied.

The quantities $\underline{\alpha}$, $\underline{\nu}$, $\underline{\beta}$, and $\underline{\gamma}$ are vectors whose components are spatial frequencies with units of cycles/m. Since spatial frequencies are related to both direction cosines and wavelength,

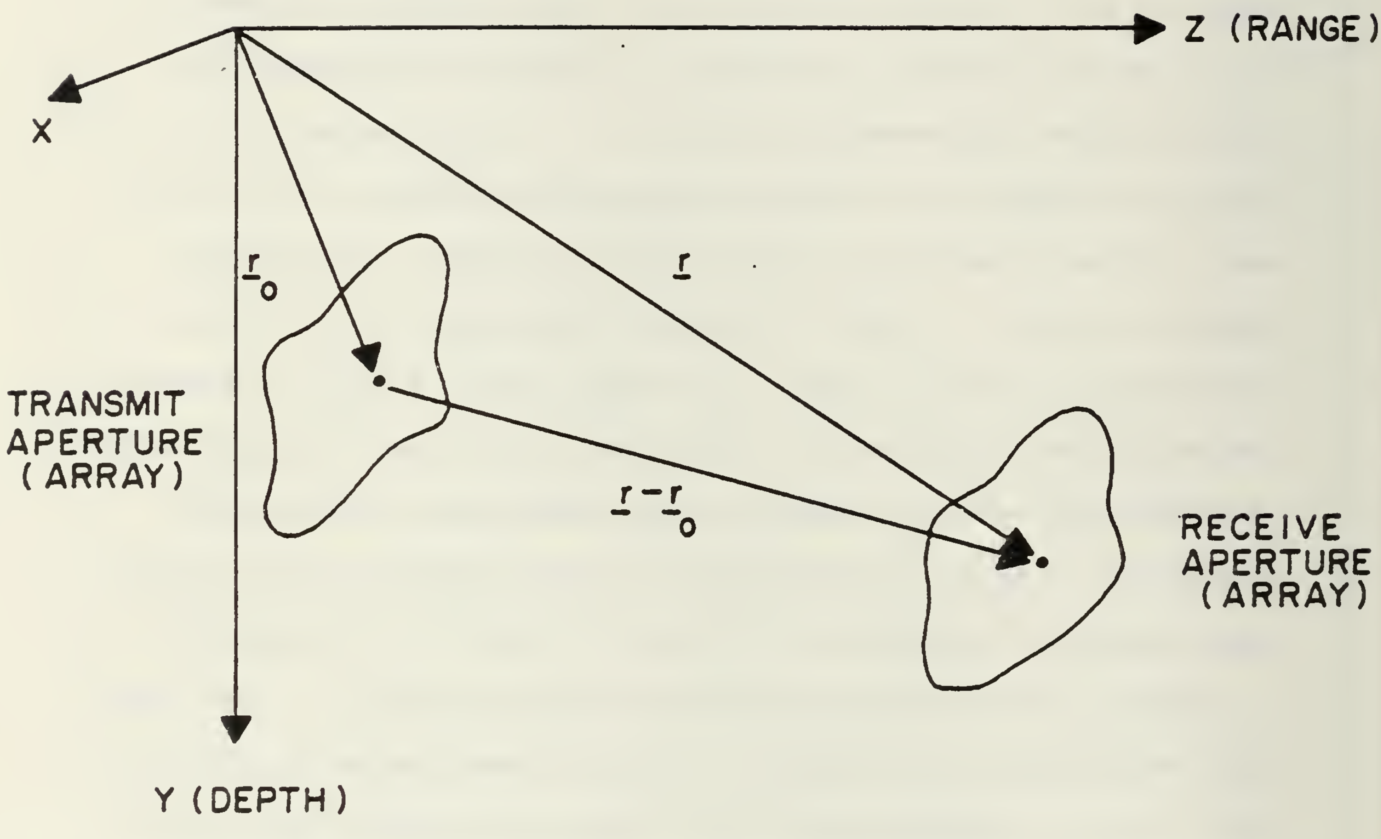

Fig. 1. Basic bistatic communication channel geometry.

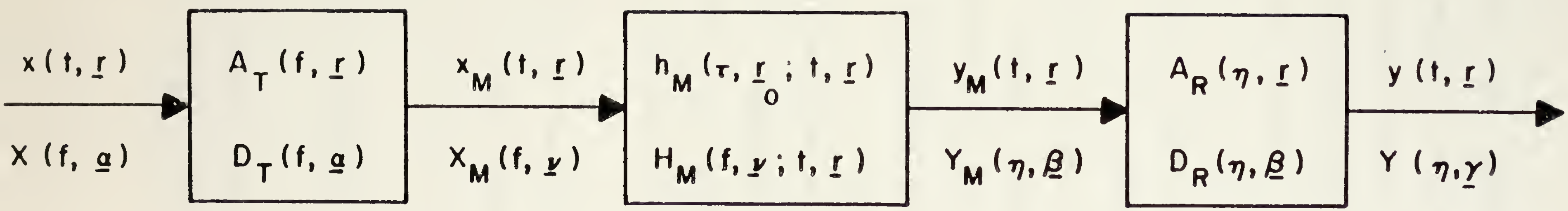

Fig. 2. Mathematical block diagram representation of the basic bistatic communication channel geometry depicted in Fig. 1.

and hence, wavenumber components, they represent directions of wave propagation. The vector $\underline{\nu}$ represents input or transmitted spatial frequencies into the medium as in $X_M(f,\underline{\nu})$, while $\underline{\beta}$ represents output or received spatial frequencies from the medium as in $Y_M(\eta,\underline{\beta})$. Note, that if $\underline{\beta} \neq \underline{\nu}$, angular spread (scatter) is implied.

The remaining expressions found in Fig. 2 are further described in the following list:

$x(t,\underline{r})$ - input electrical signal to transmit electro-acoustic transducer applied at time t and spatial location $\underline{r}$ of transducer.

$X(f,\underline{\alpha})$ - frequency (f) and angular ($\underline{\alpha}$) spectrum of input electrical signal.

$A_T(f,\underline{r})$ - complex frequency response at spatial location $\underline{r}$ of transmit transducer. Also referred to as the complex transmit aperture.

$D_T(f,\underline{\alpha})$ - transmit far-field directivity function or beam pattern.

$x_M(t,\underline{r})$ - input acoustic signal to the medium applied at time t and spatial location $\underline{r}$. Also, output acoustic signal from transmit electro-acoustic transducer.

$X_M(f,\underline{\nu})$ - frequency (f) and angular ($\underline{\nu}$) spectrum of input acoustic signal.

$h_M(\tau,\underline{r}_o; t,\underline{r})$ - time-variant, space-variant impulse response of the ocean medium. It represents the response of the medium at time t and spatial location $\underline{r}$ due to the application of an unit impulse at time $(t-\tau)$ sec., or τ sec. ago, at a distance $|\underline{r}-\underline{r}_o|$ m. away (see Fig. 1).

$H_M(f,\underline{\nu};t,\underline{r})$ - time-variant, space-variant transfer function of the ocean medium.

$y_M(t,\underline{r})$ - output acoustic signal from the medium at time t and spatial location $\underline{r}$. Also, input acoustic signal to receive electro-acoustic transducer.

$Y_M(\eta,\underline{\beta})$ - frequency (η) and angular ($\underline{\beta}$) spectrum of output acoustic signal.

$A_R(\eta,\underline{r})$ - complex frequency response at spatial location $\underline{r}$ of receive transducer. Also referred to as the complex receive aperture.

$D_R(\eta,\underline{\beta})$ - receive far-field directivity function or beam pattern.

$y(t,\underline{r})$ - output electrical signal from receive electro-acoustic transducer at time t and spatial location $\underline{r}$ of transducer.

$Y(\eta,\underline{\gamma})$ - frequency (η) and angular ($\underline{\gamma}$) spectrum of output electrical signal.

As was mentioned previously, we can represent the ocean medium as a linear, time-variant, space-variant, random filter. The term "time-variant" implies motion amongst targets, the ocean surface, discrete point scatterers, and the transmit and receive apertures (arrays). Discrete point scatterers in the ocean may include, for example, gas bubbles, fish, and other particulate matter. The time-variant property results in both Doppler spread and spread in round-trip time delay values. If the filter is time-invariant, then no motion is implied. As a result, there will be no Doppler spread and no spread in round-trip time delay.

The term "space-variant" implies that the sound speed profile (index of refraction) of the ocean is a function of position. The space-variant property results in scatter or angular spread due to refraction. If the filter is space-invariant, then an isospeed medium is implied. As a result, there will be no refraction, and hence, no scatter or angular spread since the sound rays will be travelling in straight lines.

In addition, since any motion and/or the index of refraction can be decomposed into a sum of deterministic (average) and random (fluctuating) components, these random components can be accounted for via a random filter representation vis-a-vis a deterministic filter representation.

By using a systems theory approach, surface, volume, and/or bottom reverberation returns can be modelled as the outputs from linear filters. In addition, target returns can also be modelled as filter outputs. Furthermore, different transmit signals and transmit and receive directivity functions can easily be coupled to various models (i.e., transfer functions) of the random, inhomogeneous ocean medium in a straightforward and logical fashion in order to study their effects on target detection or parameter estimation using various space-time signal processing algorithms.

The approach of treating the ocean as an isospeed medium, and hence, as a linear, time-variant, random communication channel is well established [1-19]. This linear, time-varying, random system theory approach has also been applied to target scattering problems in radar astronomy [20] and to communication channels in general [21-23]. However, with respect to target models, past research efforts have been devoted mainly to the slowly fluctuating point target problem [24-31]. Efforts to treat more complicated target models were made by Kooij [32], Moose [10], and Ziomek and Sibul [19,33]. Kooij [32] and Moose [10] both modelled the target as a linear, time-invariant, deterministic filter while Ziomek and Sibul [19,33] modelled the target as a linear, time-varying, random filter. In addition, Ziomek [34] has shown that the form of the generalized ambiguity function can be derived by treating the scattered acoustic pressure field from a point target (in relative motion with respect to a bistatic transmit/receive array geometry) as the output of a linear, time-varying, random filter.

Some work has been done in treating the ocean medium as a linear, time-variant, space-variant, random filter by Laval [9,35] and Laval and Labasque [36]. However, the notation used to incorporate the space-variant property is ad hoc, i.e., spatial variables are simply included in the arguments of the impulse response and transfer functions, for example, rather than having evolved from a systematic and consistent notation based upon linear, time-varying, space-varying system theory. In addition, Laval and Labasque [36] assume functional forms for the ocean transfer function instead of deriving them. Middleton [37,38] also studied underwater acoustic propagation in a random, inhomogeneous ocean, but did not concern himself directly with the derivation of random, time-variant, space-variant ocean transfer functions. He described the propagation phenomena using space-time operators.

In this paper we will study underwater acoustic propagation in a random, inhomogeneous ocean by treating the ocean medium as a linear, time-variant, space-variant, random filter. Section II is devoted to a discussion of the fundamentals of linear, time-variant, space-variant filters and is based upon the generalization of the results contained in Ziomek [39]. A consistent notation is developed in a systematic manner. Various system functions are introduced and important input-output relations and multidimensional (time-space) Fourier transform pairs are derived for both deterministic and random filters. To the best of the author's knowledge, the expressions presented in Section II have not appeared previously in the literature.

The equations necessary to couple the medium's transfer function to the far-field beam patterns of the transmit and receive apertures (arrays) and to the frequency spectrum of the transmitted signal are discussed in Section III.

Section 4.1 is devoted to the main problem of the paper which is the derivation of a random ocean transfer function incorporating the W.K.B. approximation. In the process of the derivation, the index of refraction is decomposed into deterministic and random components and it is assumed that the medium is "weakly scattering". The transfer function derivation was motivated by the work of Clarke [40] on the application of the W.K.B. approximation.

Section 4.2 is devoted to an example calculation of the equations for the random, output electrical signals appearing at <u>each</u> element in a receive planar array of complex weighted point sources in terms of the frequency spectrum of the transmitted electrical signal, the transmit and receive arrays, and the random ocean medium transfer function.

Finally, in Section 4.3, the autocorrelation function of the transfer function, which is also known as the coherence function, is calculated.

II. FUNDAMENTALS OF LINEAR TIME-VARIANT SPACE-VARIANT FILTERS

2.1 Deterministic Filters

2.1.1 Impulse response and transfer functions

A linear, time-variant, space-variant, filter is depicted in Fig. 3, where it is characterized by its corresponding time-varying, space-varying impulse response $h(\tau,\underline{r}_o; t,\underline{r})$. The function $h(\tau,\underline{r}_o; t,\underline{r})$ describes the response of the filter at time t and spatial location $\underline{r} = (x,y,z)$ due to the application of an unit impulse at time $(t-\tau)$, or τ seconds ago, and at a distance $|\underline{r}-\underline{r}_o|$ meters away where $\underline{r}_o = (x_o,y_o,z_o)$. Note that

$$h(\tau,\underline{r}_o;\ t,\underline{r}) \equiv h(t,\underline{r};\ t-\tau,\ \underline{r}-\underline{r}_o). \qquad (2.1\text{-}1)$$

The relationship between the input signal $x(t,\underline{r})$ and the output signal $y(t,\underline{r})$ is given by

$$y(t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\tau,\ \underline{r}-\underline{r}_o)\,h(\tau,\underline{r}_o;\ t,\underline{r})\,d\tau\,d\underline{r}_o \qquad (2.1\text{-}2)$$

where it should be noted that both the input and output signals are functions of time and space.

Example 2.1-1

Different forms of Eq. (2.1-2) can be obtained by making the following simplifying assumptions:

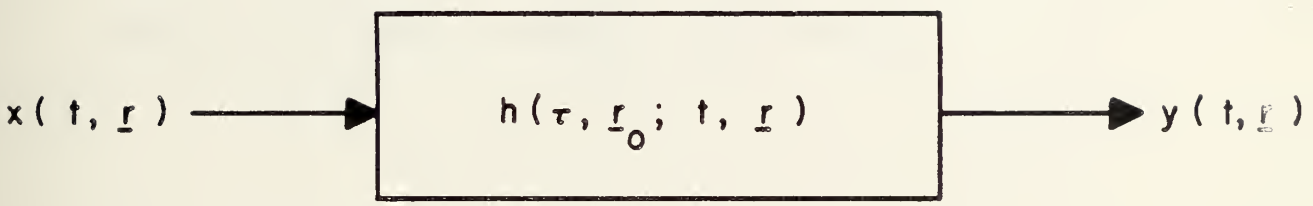

Fig. 3. Representation of a linear, time-variant, space-variant, filter.

(1) if the linear filter h is time-_in_variant and space-_in_variant, then

$$h(\tau,\underline{r}_o;\ t,\underline{r}) = h(t - [t-\tau],\ \underline{r} - [\underline{r}-\underline{r}_o])$$

$$= h(\tau,\underline{r}_o) \qquad (2.1\text{-}3)$$

and as a result, Eq. (2.1-2) reduces to

$$y(t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\tau,\ \underline{r}-\underline{r}_o)h(\tau,\underline{r}_o)d\tau d\underline{r}_o \qquad (2.1\text{-}4)$$

which is a multidimensional convolution integral as would be expected.

(2) if h is time-_in_variant but space-variant, then

$$h(\tau,\underline{r}_o;\ t,\underline{r}) = h(t - [t-\tau],\ \underline{r};\ \underline{r}-\underline{r}_o)$$

$$= h(\tau,\underline{r};\ \underline{r}-\underline{r}_o)$$

$$= h(\tau,\underline{r}_o;\ \underline{r}) \qquad (2.1\text{-}5)$$

and as a result, Eq. (2.1-2) reduces to

$$y(t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\tau,\ \underline{r}-\underline{r}_o)h(\tau,\underline{r}_o;\ \underline{r})d\tau d\underline{r}_o. \qquad (2.1\text{-}6)$$

(3) if h is time-variant but space-invariant, then

$$h(\tau,\underline{r}_o;\ t,\underline{r}) = h(t,\ \underline{r} - [\underline{r}-\underline{r}_o];\ t-\tau)$$

$$= h(t,\underline{r}_o;\ t-\tau)$$

$$= h(\tau,\underline{r}_o;\ t) \qquad (2.1\text{-}7)$$

and as a result, Eq. (2.1-2) reduces to

$$y(t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\tau,\ \underline{r}-\underline{r}_o)h(\tau,\underline{r}_o;\ t)d\tau d\underline{r}_o. \qquad (2.1\text{-}8)$$

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Note that if Eq. (2.1-2) is rewritten as

$$y(t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\alpha,\ \underline{r}-\underline{\zeta})h(\alpha,\underline{\zeta};\ t,\underline{r})d\alpha d\underline{\zeta} \qquad (2.1\text{-}9)$$

and if the input is an unit impulse applied at time $\alpha = (t-\tau)$ seconds at a distance $|\underline{\zeta}| = |\underline{r}-\underline{r}_o|$ meters away, i.e., if

$$x(\alpha,\underline{\zeta}) = \delta(\alpha - [t-\tau],\ \underline{\zeta} - [\underline{r}-\underline{r}_o]), \qquad (2.1\text{-}10)$$

then substituting Eq. (2.1-10) into Eq. (2.1-9) yields

$$y(t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\tau-\alpha,\ \underline{r}_o-\underline{\zeta})h(\alpha,\underline{\zeta};\ t,\underline{r})d\alpha d\underline{\zeta}$$

$$= h(\tau,\underline{r}_o;\ t,\underline{r}). \tag{2.1-11}$$

Analogous to the frequency response or transfer function H(f) of linear, time-invariant systems is the time-varying, space-varying, frequency response or transfer function $H(f,\underline{\nu};\ t,\underline{r})$ of linear, time-variant, space-variant, systems. It is defined as follows:

$$H(f,\underline{\nu};\ t,\underline{r}) \triangleq F_{\tau}F_{\underline{r}_o}\{h(\tau,\underline{r}_o;\ t,\underline{r})\} \tag{2.1-12}$$

or

$$H(f,\underline{\nu};\ t,\underline{r}) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau,\underline{r}_o;\ t,\underline{r})\exp(-j2\pi f\tau)\exp(+j2\pi\underline{\nu}\cdot\underline{r}_o)d\tau d\underline{r}_o \tag{2.1-13}$$

where f corresponds to input frequencies in HZ. and $\underline{\nu}$ is a vector whose components are input <u>spatial</u> <u>frequencies</u> with units of cycles/meter. As was previously mentioned in Section 1, spatial frequencies are related to wavenumber components, and hence, they represent directions of wave propagation. Similarly,

$$h(\tau,\underline{r}_o;\ t,\underline{r}) = F_f^{-1}F_{\underline{\nu}}^{-1}\{H(f,\underline{\nu};\ t,\underline{r})\} \qquad (2.1\text{-}14)$$

or

$$h(\tau,\underline{r}_o;\ t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(f,\underline{\nu};\ t,\underline{r})\exp(+j2\pi f\tau)\exp(-j2\pi\underline{\nu}\cdot\underline{r}_o)\,df\,d\underline{\nu}. \qquad (2.1\text{-}15)$$

The choice of a plus (+) sign in the exponent of $\exp(+j2\pi\underline{\nu}\cdot\underline{r}_o)$ appearing in Eq. (2.1-13), which corresponds to the forward spatial Fourier transform w.r.t. $\underline{r}_o$, was <u>not</u> arbitrary. This choice of sign convention is meant to be consistent with that of the spatial Fourier transform relationship between a complex aperture function and its directivity function (beam pattern) as is developed later in Section 3. Besides, the integrand term

$$\exp(-j2\pi f\tau)\exp(+j2\pi\underline{\nu}\cdot\underline{r}_o)$$

appearing in Eq. (2.1-13) has the nice physical interpretation of being a time-harmonic plane wave travelling in the direction of increasing $r_o = |\underline{r}_o|$ with the sign convention as given.

<u>Example 2.1-2</u>

If the linear filter h is time-<u>in</u>variant and space-<u>in</u>variant, then its corresponding transfer function can be obtained by substituting Eq. (2.1-3) into Eq. (2.1-13). Doing so yields the following result:

$$H(f,\underline{\nu}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau,\underline{r}_o)\exp(-j2\pi f\tau)\exp(+j2\pi\underline{\nu}\cdot\underline{r}_o)d\tau d\underline{r}_o \ . \tag{2.1-16}$$

Similarly, from Eq. (2.1-15),

$$h(\tau,\underline{r}_o) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(f,\underline{\nu})\exp(+j2\pi f\tau)\exp(-j2\pi\underline{\nu}\cdot\underline{r}_o)dfd\underline{\nu}. \tag{2.1-17}$$

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Calculating the transfer function according to Eq. (2.1-13) assumes that one knows the impulse response function. However, even if the impulse response is <u>not</u> known, the transfer function can still be obtained by using a time-harmonic plane wave as an input signal. This can be shown by representing a linear, time-varying, space-varying, filter by the linear operator $L(\cdot)$ which operates on input signals that are functions of both time and space. The output of the filter $y(t,\underline{r})$ can then be expressed as

$$y(t,\underline{r}) = L[x(t,\underline{r})]. \tag{2.1-18}$$

If we let

$$x(t,\underline{r}) = \exp(+j2\pi ft)\exp(-j2\pi\underline{\nu}\cdot\underline{r}), \quad (2.1\text{-}19)$$

which is in the form of a time-harmonic plane wave travelling in the direction of increasing $r = |\underline{r}|$, then from Eqs. (2.1-2) and (2.1-18) we can write

$$L[\exp(+j2\pi ft)\exp(-j2\pi\underline{\nu}\cdot\underline{r})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[+j2\pi f(t-\tau)]\exp[-j2\pi\underline{\nu}\cdot(\underline{r}-\underline{r}_o)]\cdot h(\tau,\underline{r}_o;\ t,\underline{r})d\tau d\underline{r}_o, \quad (2.1\text{-}20)$$

and by using the definition of the transfer function given by Eq. (2.1-13), Eq. (2.1-20) can be expressed as

$$L[\exp(+j2\pi ft)\exp(-j2\pi\underline{\nu}\cdot\underline{r})] = H(f,\underline{\nu};\ t,\underline{r})\exp(+j2\pi ft)\exp(-j2\pi\underline{\nu}\cdot\underline{r}) \quad (2.1\text{-}21)$$

where $L[\exp(+j2\pi ft)\exp(-j2\pi\underline{\nu}\cdot\underline{r})]$ is the response of the filter to $\exp(+j2\pi ft)\exp(-j2\pi\underline{\nu}\cdot\underline{r})$, and $H(f,\underline{\nu};\ t,\underline{r})$ is the time-varying, space-varying transfer function of the filter evaluated at f and $\underline{\nu}$.

Equation (2.1-21) is a fundamental result that will be used in the derivation of an ocean transfer function in Section 4.1. In addition, Eq. (2.1-21) can be used to represent the output $y(t,\underline{r})$ in terms of the transfer function. For example, if we express the input $x(t,\underline{r})$ in terms of the following multidimensional inverse Fourier transform

$$x(t,\underline{r}) = F_f^{-1}F_{\underline{\nu}}^{-1}\{X(f,\underline{\nu})\} \tag{2.1-22}$$

or

$$x(t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f,\underline{\nu})\exp(+j2\pi ft)\exp(-j2\pi\underline{\nu}\cdot\underline{r})\,df\,d\underline{\nu}, \tag{2.1-23}$$

then from Eqs. (2.1-18) and (2.1-21), we can write the output $y(t,\underline{r})$ as

$$y(t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f,\underline{\nu})L[\exp(+j2\pi ft)\exp(-j2\pi\underline{\nu}\cdot\underline{r})]\,df\,d\underline{\nu}. \tag{2.1-24}$$

or

$$y(t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f,\underline{\nu})H(f,\underline{\nu};\ t,\underline{r})\exp(+j2\pi ft)\exp(-j2\pi\underline{\nu}\cdot\underline{r})\,df\,d\underline{\nu} \tag{2.1-25}$$

Compare Eq. (2.1-25) with Eq. (2.1-2).

2.1.2 Spreading and bi-frequency functions

Two additional filter functions will now be introduced; namely, the spreading function and the bi-frequency function. The spreading function will be discussed first.

The spreading function $S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})$ is defined as

$$S(\tau,\underline{r}_o;\ \phi,\underline{\kappa}) \triangleq F_t F_{\underline{r}}\{h(\tau,\underline{r}_o;\ t,\underline{r})\} \qquad (2.1\text{-}26)$$

or

$$S(\tau,\underline{r}_o;\ \phi,\underline{\kappa}) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau,\underline{r}_o;\ t,\underline{r})\exp(-j2\pi\phi t)\exp(+j2\pi\underline{\kappa}\cdot\underline{r})dtd\underline{r} \qquad (2.1\text{-}27)$$

where ϕ corresponds to the rate of change of the filter's impulse response in HZ. and $\underline{\kappa}$ is a vector whose components are spatial frequencies which correspond to the rate of change of the impulse response in cycles/m. Similarly,

$$h(\tau,\underline{r}_o;\ t,\underline{r}) = F_{\phi}^{-1}F_{\underline{\kappa}}^{-1}\{S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})\} \qquad (2.1\text{-}28)$$

or

$$h(\tau,\underline{r}_o;\ t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})\exp(+j2\pi\phi t)\exp(-j2\pi\underline{\kappa}\cdot\underline{r})d\phi d\underline{\kappa}. \qquad (2.1\text{-}29)$$

An alternate representation of the output $y(t,\underline{r})$ can be obtained by substituting Eq. (2.1-29) into Eq. (2.1-2). Doing so yields

$$y(t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\tau,\ \underline{r}-\underline{r}_o)\exp(+j2\pi\phi t)\exp(-j2\pi\underline{\kappa}\cdot\underline{r})\cdot S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})\,d\phi\, d\underline{\kappa}\, d\tau\, d\underline{r}_o \qquad (2.1\text{-}30)$$

where the integrand term

$$S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})x(t-\tau,\ \underline{r}-\underline{r}_o)\exp(+j2\pi\phi t)\exp(-j2\pi\underline{\kappa}\cdot\underline{r})$$

is a time, space, and "frequency" (frequency in HZ. and spatial frequencies in cycles/m.) shifted replica of the input signal $x(t,\underline{r})$ weighted by the spreading function $S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})$. Therefore, for a given input (source) location as specified by $\underline{r}_o$, the spreading function determines the amount of spread in round-trip time delay τ (sec.), frequency ϕ (HZ.), and spatial frequencies $\underline{\kappa}$ (cycles/m.) that an input signal will undergo as it passes through a linear, time-varying, space-varying, channel. Equations (2.1-30) and (2.1-2) both indicate that the output $y(t,\underline{r})$ at some receiver location $\underline{r}$ is dependent upon the location $\underline{r}_o$ of the input (source) via the spreading function $S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})$ w.r.t. Eq. (2.1-30), and via the impulse response $h(\tau,\underline{r}_o;\ t,\underline{r})$ w.r.t. Eq. (2.1-2). For example, the existence and extent of SOFAR transmission depends on both the depths of the source and receiver [41].

The last filter function to be discussed is the bi-frequency function. The bi-frequency function $B(f,\underline{\nu};\ \phi,\underline{\kappa})$ is defined as

$$B(f,\underline{\nu};\ \phi,\underline{\kappa}) \triangleq F_t F_{\underline{r}}\{H(f,\underline{\nu};\ t,\underline{r})\} \qquad (2.1\text{-}31)$$

or

$$B(f,\underline{\nu};\ \phi,\underline{\kappa}) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(f,\underline{\nu};\ t,\underline{r})\exp(-j2\pi\phi t)\exp(+j2\pi\underline{\kappa}\cdot\underline{r})\,dt\,d\underline{r}. \qquad (2.1\text{-}32)$$

Similarly,

$$H(f,\underline{\nu};\ t,\underline{r}) = F_{\phi}^{-1} F_{\underline{\kappa}}^{-1}\{B(f,\underline{\nu};\ \phi,\underline{\kappa})\} \qquad (2.1\text{-}33)$$

or

$$H(f,\underline{\nu};\ t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(f,\underline{\nu};\ \phi,\underline{\kappa})\exp(+j2\pi\phi t)\exp(-j2\pi\underline{\kappa}\cdot\underline{r})\,d\phi\,d\underline{\kappa}. \qquad (2.1\text{-}34)$$

The term bi-frequency function was originally used to denote the appearance of the two frequency variables f and ϕ in linear, time-varying systems theory. We can generalize this notion by using the term bi-frequency function to denote the appearance of the input "frequency" pair f and $\underline{\nu}$, and the filter variation "frequency" pair ϕ and $\underline{\kappa}$ in linear, time-varying, space-varying, systems theory.

Just as the spreading function gives an indication of how rapidly the impulse response changes with time and space, the bi-frequency function gives an indication of how rapidly the transfer function changes with time and space.

The bi-frequency function can also be obtained by taking the Fourier transform of the spreading function w.r.t. τ and $\underline{r}_o$, i.e.,

$$B(f,\underline{\nu};\ \phi,\underline{\kappa}) = F_{\tau}F_{\underline{r}_o}\{S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})\} \tag{2.1-35}$$

or

$$B(f,\underline{\nu};\ \phi,\underline{\kappa}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})\exp(-j2\pi f\tau)\exp(+j2\pi\underline{\nu}\cdot\underline{r}_o)\,d\tau\,d\underline{r}_o \tag{2.1-36}$$

and, similarly,

$$S(\tau,\underline{r}_o;\ \phi,\underline{\kappa}) = F_f^{-1}F_{\underline{\nu}}^{-1}\{B(f,\underline{\nu};\ \phi,\underline{\kappa})\} \tag{2.1-37}$$

or

$$S(\tau,\underline{r}_o;\ \phi,\underline{\kappa}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(f,\underline{\nu};\ \phi,\underline{\kappa})\exp(+j2\pi f\tau)\exp(-j2\pi\underline{\nu}\cdot\underline{r}_o)\,df\,d\underline{\nu}\ . \tag{2.1-38}$$

In addition, the impulse response and bi-frequency functions form a Fourier transform pair, i.e.,

$$B(f,\underline{\nu};\ \phi,\underline{\kappa}) = F_{\tau}F_{\underline{r}_o}F_{t}F_{\underline{r}}\{h(\tau,\underline{r}_o;\ t,\underline{r})\} \quad (2.1\text{-}39)$$

or

$$B(f,\underline{\nu};\ \phi,\underline{\kappa}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau,\underline{r}_o;\ t,\underline{r})\exp(-j2\pi f\tau)\exp(+j2\pi\underline{\nu}\cdot\underline{r}_o)\cdot \exp(-j2\pi\phi t)\exp(+j2\pi\underline{\kappa}\cdot\underline{r})d\tau d\underline{r}_o dt d\underline{r}$$

(2.1-40)

and, similarly,

$$h(\tau,\underline{r}_o;\ t,\underline{r}) = F_{f}^{-1}F_{\underline{\nu}}^{-1}F_{\phi}^{-1}F_{\underline{\kappa}}^{-1}\{B(f,\underline{\nu};\ \phi,\underline{\kappa})\}$$

(2.1-41)

or

$$h(\tau,\underline{r}_o;\ t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(f,\underline{\nu};\ \phi,\underline{\kappa})\exp(+j2\pi f\tau)\exp(-j2\pi\underline{\nu}\cdot\underline{r}_o)\cdot \exp(+j2\pi\phi t)\exp(-j2\pi\underline{\kappa}\cdot\underline{r})df d\underline{\nu} d\phi d\underline{\kappa}.$$

(2.1-42)

The interdependence which exists amongst the four filter functions is illustrated in Fig. 4. Any one of these functions may be used to define completely a linear, time-varying, space-varying, communication channel.

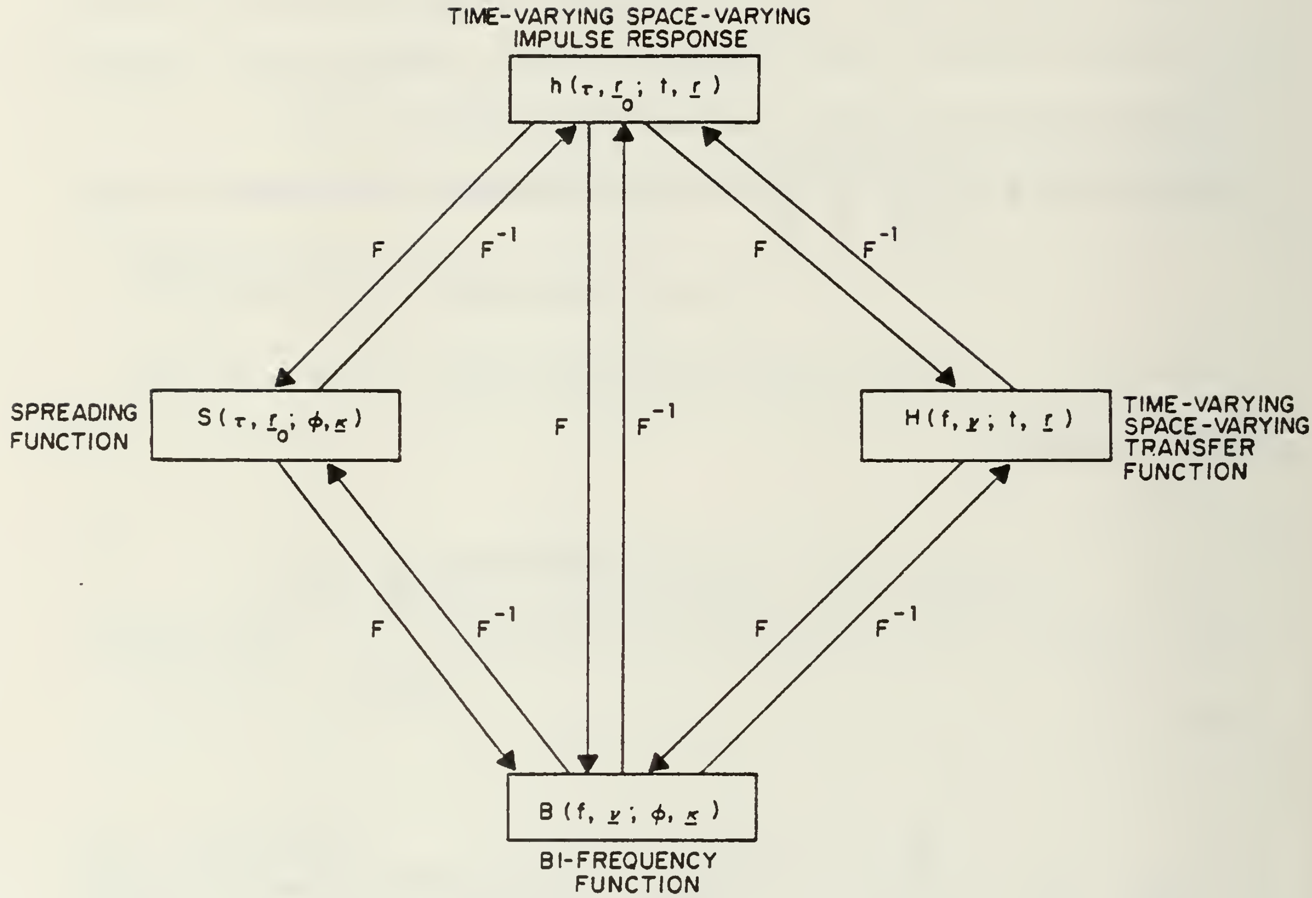

Fig. 4. Interdependence amongst the four filter functions that characterize linear, time-varying, space-varying, communication channels.

2.1.3 Output frequency and angular spectrum

The output frequency (η) and angular ($\underline{\beta}$) spectrum $Y(\eta,\underline{\beta})$ is defined as

$$Y(\eta,\underline{\beta}) \triangleq F_t F_{\underline{r}}\{y(t,\underline{r})\} \tag{2.1-43}$$

or

$$Y(\eta,\underline{\beta}) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y(t,\underline{r})\exp(-j2\pi\eta t)\exp(+j2\pi\underline{\beta}\cdot\underline{r})\,dt\,d\underline{r} \tag{2.1-44}$$

where η corresponds to output frequencies in HZ. and $\underline{\beta}$ is a vector whose components are output spatial frequencies with units of cycles/meter. Substituting Eq. (2.1-25) into Eq. (2.1-44) yields

$$Y(\eta,\underline{\beta}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f,\underline{\nu}) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(f,\underline{\nu};\ t,\underline{r})\exp[-j2\pi(\eta-f)t]\cdot \exp[+j2\pi(\underline{\beta}-\underline{\nu})\cdot\underline{r}]\,dt\,d\underline{r}\,df\,d\underline{\nu} \tag{2.1-45}$$

where, from Eq. (2.1-32), it can be seen that the inner multi-dimensional integral is equal to $B(f,\underline{\nu};\ \eta-f,\ \underline{\beta}-\underline{\nu})$ so that

$$Y(\eta,\underline{\beta}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f,\underline{\nu})B(f,\underline{\nu};\ \eta-f,\ \underline{\beta}-\underline{\nu})\,df\,d\underline{\nu} \tag{2.1-46}$$

which is in the form of a multidimensional "frequency" domain convolution integral. Note that $\phi \equiv \eta - f$ or $\eta \equiv f + \phi$, and that $\underline{\kappa} \equiv \underline{\beta} - \underline{\nu}$ or $\underline{\beta} \equiv \underline{\nu} + \underline{\kappa}$. Thus, the output frequencies η in HZ. are equal to the sum of the input frequencies f and the variations in frequency, ϕ, due to the time-varying property of the filter. Similarly, the output spatial frequencies $\underline{\beta}$ (directions of wave propagation) in cycles/m. are equal to the sum of the input spatial frequencies $\underline{\nu}$ (directions of propagation of transmitted plane waves) and the variations in spatial frequency (directions of wave propagation), $\underline{\kappa}$, due to the space-varying property of the filter. Equation (2.1-46) demonstrates that a linear, time-variant, space-variant, filter will spread the input frequency and angular spectrum $X(f,\underline{\nu})$ in both frequencies in HZ. and spatial frequencies in cycles/m.

Example 2.1-3

If the linear filter h is time-invariant and space-invariant, then $H(f,\underline{\nu};\ t,\underline{r}) = H(f,\underline{\nu})$ [see Eq. (2.1-16)] and, as a result, Eq. (2.1-32) reduces to

$$B(f,\underline{\nu};\ \phi,\underline{\kappa}) = H(f,\underline{\nu}) \int_{-\infty}^{\infty} \exp(-j2\pi\phi t)\,dt \int_{-\infty}^{\infty} \exp(+j2\pi\underline{\kappa}\cdot\underline{r})\,d\underline{r}$$

or

$$B(f,\underline{\nu};\ \phi,\underline{\kappa}) = H(f,\underline{\nu})\,\delta(\phi)\,\delta(\underline{\kappa}) \qquad (2.1\text{-}47)$$

since

$$F_t\{1\} = \delta(\phi)$$

and

$$F_{\underline{r}}\{1\} = \delta(\underline{\kappa}).$$

Substituting Eq. (2.1-47) into Eq. (2.1-46) yields

$$Y(\eta,\underline{\beta}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f,\underline{\nu})H(f,\underline{\nu})\delta(\eta-f)\delta(\underline{\beta}-\underline{\nu})dfd\underline{\nu}$$

which simplifies to

$$Y(\eta,\underline{\beta}) = X(\eta,\underline{\beta})H(\eta,\underline{\beta}),$$

and by replacing η with f and $\underline{\beta}$ with $\underline{\nu}$, we finally obtain

$$Y(f,\underline{\nu}) = X(f,\underline{\nu})H(f,\underline{\nu}) \qquad (2.1\text{-}48)$$

which is the output frequency and angular spectrum from a linear, time-<u>in</u>variant, space-<u>in</u>variant, filter. Note that the output frequency f and the output spatial frequency vector $\underline{\nu}$ are identical with those of the input $X(f,\underline{\nu})$. Hence, as would be expected, there is <u>no</u> "frequency" spreading. This would correspond to the physical situation of transmitting a signal via a transmit aperture (array) to a receive aperture (array) when the platforms containing the apertures (arrays) are <u>not</u> in motion, and the intervening ocean medium has a <u>constant</u> speed of sound (index of refraction) and no discrete point scatterers in motion. No motion implies no Doppler (frequency) spread and a constant speed of sound implies that the sound rays travel in straight lines, i.e., there is no angular spread (scatter) or change in the direction of propagation of the transmitted sound rays.

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2.2 Random Filters

2.2.1 Filter autocorrelation functions

In Section 2.1, four system functions were introduced which are used to characterize linear, time-variant, space-variant, filters. However, if the filter is random, then each of these system functions must be considered as random functions. As a result, we must work with the respective system autocorrelation functions which are defined as follows:

$$R_h(\tau,\tau',\underline{r}_o,\underline{r}_o'; t,t',\underline{r},\underline{r}') \triangleq E\{h(\tau,\underline{r}_o; t,\underline{r})h^*(\tau',\underline{r}_o'; t',\underline{r}')\} \tag{2.2-1}$$

$$R_H(f,f',\underline{\nu},\underline{\nu}'; t,t',\underline{r},\underline{r}') \triangleq E\{H(f,\underline{\nu};t,\underline{r})H^*(f',\underline{\nu}'; t',\underline{r}')\} \tag{2.2-2}$$

$$R_S(\tau,\tau',\underline{r}_o,\underline{r}_o'; \phi,\phi',\underline{\kappa},\underline{\kappa}') \triangleq E\{S(\tau,\underline{r}_o; \phi,\underline{\kappa})S^*(\tau',\underline{r}_o'; \phi',\underline{\kappa}')\} \tag{2.2-3}$$

and

$$R_B(f,f',\underline{\nu},\underline{\nu}'; \phi,\phi',\underline{\kappa},\underline{\kappa}') \triangleq E\{B(f,\underline{\nu}; \phi,\underline{\kappa})B^*(f',\underline{\nu}'; \phi',\underline{\kappa}')\} \tag{2.2-4}$$

where $E\{\cdot\}$ is the expectation or ensemble average operator and the asterisk denotes complex conjugate.

If we use the sign convention that forward transforms w.r.t. $\tau,t,\underline{r}_o'$, and $\underline{r}'$ are defined with a minus sign in the exponent of the complex exponential (inverse transforms w.r.t. $f, \phi, \underline{\nu}'$, and $\underline{\kappa}'$ are defined with a plus sign) and forward transforms w.r.t. $\tau', t', \underline{r}_o$, and $\underline{r}$ are defined with a plus sign (inverse transforms w.r.t. $f', \phi', \underline{\nu}$, and $\underline{\kappa}$ are defined with a minus sign), then it can be shown that the four system autocorrelation functions are related by the following Fourier transform pairs:

$$R_H(f,f',\underline{\nu},\underline{\nu}';\ t,t',\underline{r},\underline{r}') = F_\tau F_{\tau'} F_{\underline{r}_o} F_{\underline{r}_o'}\{R_h(\tau,\tau',\underline{r}_o,\underline{r}_o';\ t,t',\underline{r},\underline{r}') \tag{2.2-5}$$

and

$$R_h(\tau,\tau',\underline{r}_o,\underline{r}_o';\ t,t',\underline{r},\underline{r}') = F_f^{-1} F_{f'}^{-1} F_{\underline{\nu}}^{-1} F_{\underline{\nu}'}^{-1}\{R_H(f,f',\underline{\nu},\underline{\nu}';\ t,t',\underline{r},\underline{r}') \tag{2.2-6}$$

$$R_B(f,f',\underline{\nu},\underline{\nu}';\ \phi,\phi',\underline{\kappa},\underline{\kappa}') = F_t F_{t'} F_{\underline{r}} F_{\underline{r}'}\{R_H(f,f',\underline{\nu},\underline{\nu}';\ t,t',\underline{r},\underline{r}')\} \tag{2.2-7}$$

and

$$R_H(f,f',\underline{\nu},\underline{\nu}';t,t',\underline{r},\underline{r}') = F_\phi^{-1} F_{\phi'}^{-1} F_{\underline{\kappa}}^{-1} F_{\underline{\kappa}'}^{-1}\{R_B(f,f',\underline{\nu},\underline{\nu}';\ \phi,\phi',\underline{\kappa},\underline{\kappa}')\}; \tag{2.2-8}$$

$$R_S(\tau,\tau',\underline{r}_o,\underline{r}_o';\ \phi,\phi',\underline{\kappa},\underline{\kappa}') = F_t F_{t'} F_{\underline{r}} F_{\underline{r}'}\{R_h(\tau,\tau',\underline{r}_o,\underline{r}_o';\ t,t',\underline{r},\underline{r}')\} \tag{2.2-9}$$

and

$$R_h(\tau,\tau',\underline{r}_o,\underline{r}_o';\ t,t',\underline{r},\underline{r}') = F_\phi^{-1} F_{\phi'}^{-1} F_{\underline{\kappa}}^{-1} F_{\underline{\kappa}'}^{-1}\{R_S(\tau,\tau',\underline{r}_o,\underline{r}_o';\ \phi,\phi',\underline{\kappa},\underline{\kappa}' \tag{2.2-10}$$

$$R_B(f,f',\underline{\nu},\underline{\nu}';\ \phi\ \phi',\underline{\kappa},\underline{\kappa}') = F_\tau F_{\tau'} F_{\underline{r}_o} F_{\underline{r}_o'}\{R_S(\tau,\tau',\underline{r}_o,\underline{r}_o';\ \phi,\phi',\underline{\kappa},\underline{\kappa}')\} \tag{2.2-11}$$

and

$$R_S(\tau,\tau',\underline{r}_o,\underline{r}_o';\ \phi,\phi',\underline{\kappa},\underline{\kappa}') = F_f^{-1}F_{f'}^{-1}F_{\underline{\nu}}^{-1}F_{\underline{\nu}'}^{-1}\{R_B(f,f',\underline{\nu},\underline{\nu}';\ \phi,\phi',\underline{\kappa},\underline{\kappa}')\}; \tag{2.2-12}$$

and finally,

$$R_B(f,f',\underline{\nu},\underline{\nu}';\ \phi,\phi',\underline{\kappa},\underline{\kappa}') = F_\tau F_{\tau'} F_{\underline{r}_o} F_{\underline{r}_o'} F_t F_{t'} F_{\underline{r}} F_{\underline{r}'}\{R_h(\tau,\tau',\underline{r}_o,\underline{r}_o';\ t,t'\underline{r},\underline{r}')\} \tag{2.2-13}$$

and

$$R_h(\tau,\tau',\underline{r}_o,\underline{r}_o';\ t,t',\underline{r},\underline{r}') = F_f^{-1}F_{f'}^{-1}F_{\underline{\nu}}^{-1}F_{\underline{\nu}'}^{-1}F_\phi^{-1}F_{\phi'}^{-1}F_{\underline{\kappa}}^{-1}F_{\underline{\kappa}'}^{-1}\{R_B(f,f',\underline{\nu},\underline{\nu}';\ \phi,\phi',\underline{\kappa},\underline{\kappa}')\}. \tag{2.2-14}$$

The interdependence which exists amongst the four filter autocorrelation functions is illustrated in Fig. 5.

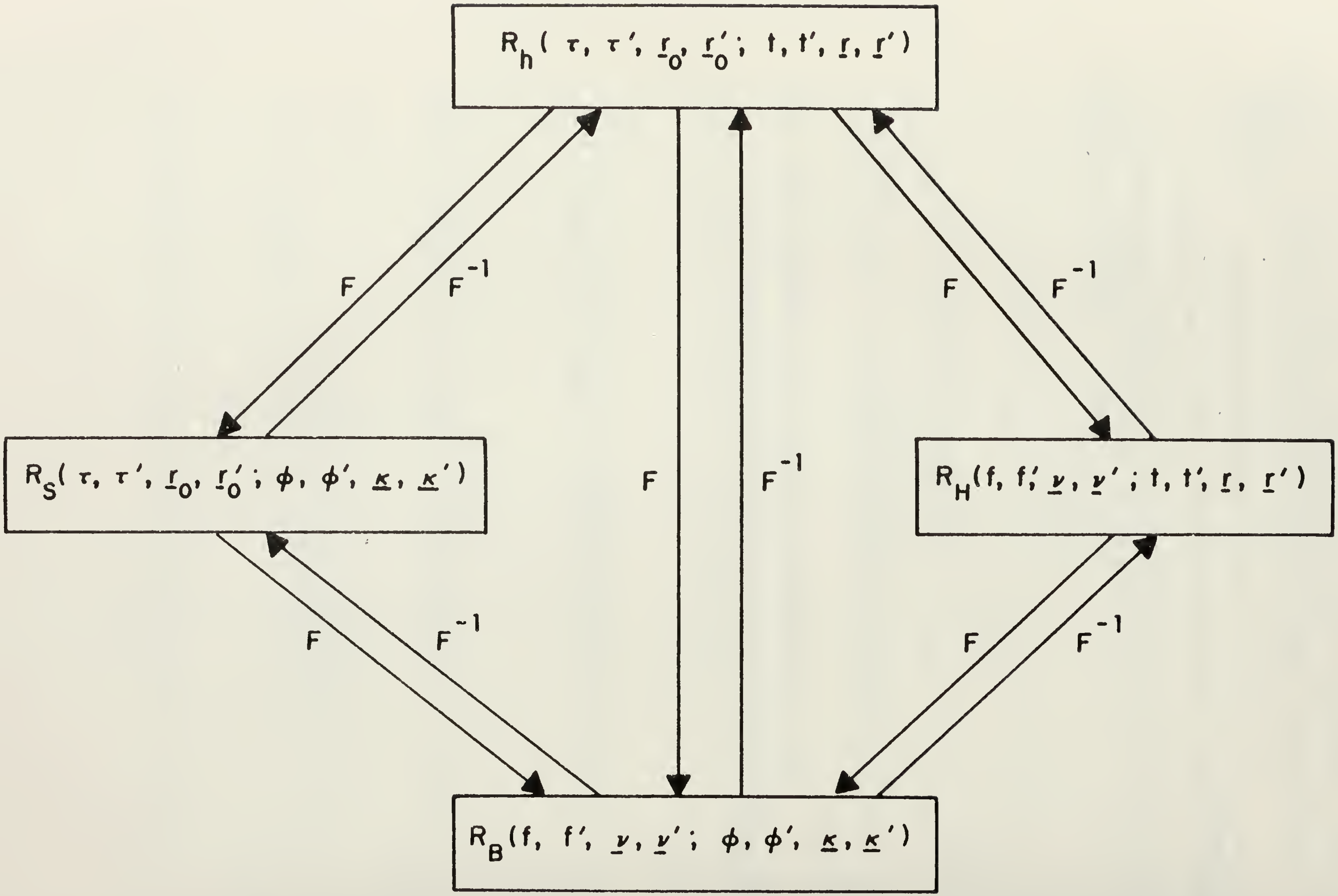

Fig. 5. Interdependence amongst the four filter autocorrelation functions that characterize linear, time-varying, space-varying, random communication channels.

2.2.2 Uncorrelated spreading and the scattering function

In this section we will examine the consequences of assuming that the spreading function $S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})$ is uncorrelated with $S(\tau',\ \underline{r}'_o;\ \phi',\underline{\kappa}')$ for all values of $\tau' \neq \tau$, $\underline{r}'_o \neq \underline{r}_o$, $\phi' \neq \phi$, and $\underline{\kappa}' \neq \underline{\kappa}$.

The assumption of uncorrelated spreading is mathematically equivalent to stating that the autocovariance of $S(\tau,\underline{r}_o;\phi,\underline{\kappa})$ and $S(\tau',\underline{r}'_o;\ \phi',\underline{\kappa}')$ is zero for all values of $\tau' \neq \tau$, $\underline{r}'_o \neq \underline{r}_o$, $\phi' \neq \phi$, and $\underline{\kappa}' \neq \underline{\kappa}$, i.e.,

$$C_S(\tau,\tau',\underline{r}_o,\underline{r}'_o;\phi,\phi',\underline{\kappa},\underline{\kappa}') = R_S(\tau,\tau',\underline{r}_o,\underline{r}'_o;\ \phi,\phi',\underline{\kappa},\underline{\kappa}') - \mu_S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})\ \mu_S^*(\tau',\underline{r}'_o;\ \phi',\underline{\kappa}') = 0;$$
$$\tau' \neq \tau,\ \underline{r}'_o \neq \underline{r}_o,\ \phi' \neq \phi, \text{ and } \underline{\kappa}' \neq \underline{\kappa}. \qquad (2.2\text{-}15)$$

where C_S is the autocovariance function and $\mu_S(\tau,\underline{r}_o;\ \phi,\underline{\kappa}) = E\{S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})\}$. If it is assumed that $\mu_S(\tau,\underline{r}_o;\ \phi,\underline{\kappa}) = 0$, then Eq. (2.2-15) is equivalent to

$$R_S(\tau,\tau',\underline{r}_o,\underline{r}'_o;\phi,\phi',\underline{\kappa},\underline{\kappa}') = R_S(\tau,\underline{r}_o;\phi,\underline{\kappa})\,\delta(\tau-\tau')\cdot \delta(\underline{r}_o-\underline{r}'_o)\,\delta(\phi-\phi')\,\delta(\underline{\kappa}-\underline{\kappa}') \qquad (2.2\text{-}16)$$

where

$$R_S(\tau,\underline{r}_o;\phi,\underline{\kappa}) = E\{|S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})|^2\} \qquad (2.2\text{-}17)$$

is called the <u>scattering function</u> and is equal to the mean squared value of the spreading function. The scattering function can be thought of as an average power density function which determines the average amount of spread that an input signal's power will undergo as a function of round-trip time delay τ, frequency ϕ, and spatial frequencies $\underline{\kappa}$ for a given input (source) location $\underline{r}_o$. Note that the scattering function is a real, positive valued function.

Equation (2.2-16) is the result of the assumption that the spreading function is zero mean. However, if the spreading function is non-zero mean, it is convenient to do the analysis with the <u>centered process</u>

$$S_C(\tau,\underline{r}_o;\ \phi,\underline{\kappa}) = S(\tau,\underline{r}_o;\ \phi,\underline{\kappa}) - \mu_S(\tau,\underline{r}_o;\ \phi,\underline{\kappa}).$$

The random process $S_C(\tau,\underline{r}_o;\ \phi,\underline{\kappa})$ has zero mean, and as a result, its autocovariance function is equal to

$$\begin{aligned} C_{S_C}(\tau,\tau',\underline{r}_o,\underline{r}_o';\ \phi,\phi',\underline{\kappa},\underline{\kappa}') &= E\{S_C(\tau,\underline{r}_o;\ \phi,\underline{\kappa})S_C^*(\tau',\underline{r}_o';\ \phi',\underline{\kappa}')\} \\ &= R_{S_C}(\tau,\tau',\underline{r}_o,\underline{r}_o';\ \phi,\phi',\underline{\kappa},\underline{\kappa}') \\ &= C_S(\tau,\tau',\underline{r}_o,\underline{r}_o';\ \phi,\phi',\underline{\kappa},\underline{\kappa}'), \end{aligned}$$

i.e., the autocorrelation function of the zero mean centered process is equal to the autocovariance function of the original non-zero mean spreading function.

Let us next examine the effect that the assumption of uncorrelated spreading has on the remaining three system autocorrelation functions. As Fig. 5 indicates, R_H can be obtained from R_S by performing the following transformations:

$$R_H(f,f',\underline{\nu},\underline{\nu}'\,;\,t,t',\underline{r},\underline{r}') = F_{\phi}^{-1}F_{\phi'}^{-1}F_{\underline{\kappa}}^{-1}F_{\underline{\kappa}'}^{-1}F_{\tau}F_{\tau'}F_{\underline{r}_o}F_{\underline{r}_o'}\{R_S(\tau,\tau',\underline{r}_o,\underline{r}_o';\ \phi,\phi',\underline{\kappa},\underline{\kappa}')\} \qquad (2.2\text{-}18)$$

or

$$R_H(f,f',\underline{\nu},\underline{\nu}'\,;\,t,t',\underline{r},\underline{r}') = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} R_S(\tau,\tau',\underline{r}_o,\underline{r}_o';\ \phi,\phi',\underline{\kappa},\underline{\kappa}')\cdot \exp[+j2\pi(\phi t-\phi' t')]\exp[-j2\pi(\underline{\kappa}\cdot\underline{r}-\underline{\kappa}'\cdot\underline{r}')]\cdot \exp[-j2\pi(f\tau-f'\tau')]\exp[+j2\pi(\underline{\nu}\cdot\underline{r}_o-\underline{\nu}'\cdot\underline{r}_o')]\cdot d\phi d\phi' d\underline{\kappa} d\underline{\kappa}' d\tau d\tau' d\underline{r}_o d\underline{r}_o' \ . \qquad (2.2\text{-}19)$$

If Eq. (2.2-16) is substituted into Eq. (2.2-19), then

$$R_H(f,f',\underline{\nu},\underline{\nu}';\ t,t',\underline{r},\underline{r}') = R_H(\Delta f,\Delta\underline{\nu};\ \Delta t,\Delta\underline{r}) \qquad (2.2\text{-}20)$$

where

$$R_H(\Delta f,\Delta\underline{\nu};\ \Delta t,\Delta\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})\exp(+j2\pi\phi\Delta t)\cdot \exp(-j2\pi\underline{\kappa}\cdot\Delta\underline{r})\exp(-j2\pi\Delta f\tau)\cdot \exp(+j2\pi\Delta\underline{\nu}\cdot\underline{r}_o)\,d\phi d\underline{\kappa} d\tau d\underline{r}_o ,$$

$$(2.2\text{-}21)$$

$$\Delta f = f-f', \ \Delta\underline{\nu} = \underline{\nu}-\underline{\nu}', \ \Delta t = t-t', \text{ and } \Delta\underline{r} = \underline{r}-\underline{r}'.$$

It can be seen from Eq. (2.2-20) that when uncorrelated spreading is assumed, the autocorrelation function R_H becomes a function of the differences Δf, $\Delta\underline{\nu}$, Δt, and $\Delta\underline{r}$ only. This implies that the random process $H(f,\underline{\nu};\ t,\underline{r})$ is wide-sense stationary in frequency, spatial frequencies, time, and space. An additional requirement for $H(f,\underline{\nu};\ t,\underline{r})$ to be wide-sense stationary is that the mean value

$$\mu_H(f,\underline{\nu};\ t,\underline{r}) = E\{H(f,\underline{\nu};\ t,\underline{r})\}$$

is a constant. Since the four filter functions are related by linear transformations (see Fig. 4), and since it was assumed that $\mu_S(\tau,\underline{r}_o;\ \phi,\underline{\kappa}) = 0$, then $\mu_h(\tau,\underline{r}_o;\ t,\underline{r}) = 0$, $\mu_H(f,\underline{\nu};\ t,\underline{r}) = 0$, and $\mu_B(f,\underline{\nu};\ \phi,\underline{\kappa}) = 0$ which are constant, zero mean values. Therefore, the condition of uncorrelated spreading in round-trip time delay τ, input (source) location $\underline{r}_o$, frequency ϕ, and spatial frequencies $\underline{\kappa}$ is equivalent to a condition of wide-sense stationarity in frequency Δf, spatial frequencies $\Delta\underline{\nu}$, time Δt, and output (receiver) location $\Delta\underline{r}$, respectively. If uncorrelated spreading in τ, $\underline{r}_o$, ϕ, and $\underline{\kappa}$ occur together, then we have a <u>wide-sense stationary uncorrelated spreading</u> (WSSUS) communication channel.

Consider the autocorrelation function R_h next. If Eq. (2.2-16) is substituted into Eq. (2.2-10) and the indicated transformations are performed, then

$$R_h(\tau,\tau',\underline{r}_o,\underline{r}_o';\ t,t',\underline{r},\underline{r}') = R_h(\tau,\underline{r}_o;\ \Delta t,\Delta\underline{r})\,\delta(\tau-\tau')\,\delta(\underline{r}_o-\underline{r}_o') \tag{2.2-22}$$

where

$$R_h(\tau,\underline{r}_o;\ \Delta t,\Delta\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})\exp(+j2\pi\phi\Delta t)\cdot \exp(-j2\pi\underline{\kappa}\cdot\Delta\underline{r})\,d\phi\, d\underline{\kappa}. \tag{2.2-23}$$

Equation (2.2-22) indicates that the random process $h(\tau,\underline{r}_o;\ t,\underline{r})$ is wide-sense stationary in time and space because of the Δt and $\Delta\underline{r}$ dependence and since $\mu_h(\tau,\underline{r}_o;\ t,\underline{r}) = 0$. Equation (2.2-22) also indicates that $h(\tau,\underline{r}_o;\ t,\underline{r})$ is uncorrelated for all values of $\tau' \neq \tau$ and $\underline{r}_o' \neq \underline{r}_o$.

Finally, if Eq. (2.2-16) is substituted into Eq. (2.2-11) and the indicated transformations are performed, then

$$R_B(f,f',\underline{\nu},\underline{\nu}';\ \phi,\phi',\underline{\kappa},\underline{\kappa}') = R_B(\Delta f,\Delta\underline{\nu};\phi,\underline{\kappa})\,\delta(\phi-\phi')\,\delta(\underline{\kappa}-\underline{\kappa}') \tag{2.2-24}$$

where

$$R_B(\Delta f,\Delta\underline{\nu};\ \phi,\underline{\kappa}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})\exp(-j2\pi\Delta f\tau)\cdot \exp(+j2\pi\Delta\underline{\nu}\cdot\underline{r}_o)\,d\tau\, d\underline{r}_o. \tag{2.2-25}$$

Equation (2.2-24) indicates that the random process $B(f,\underline{\nu};\ \phi,\underline{\kappa})$ is wide-sense stationary in frequency and spatial frequencies because of the Δf and $\Delta\underline{\nu}$ dependence and since $\mu_B(f,\underline{\nu};\ \phi,\underline{\kappa}) = 0$. Equation (2.2-24) also indicates that $B(f,\underline{\nu};\ \phi,\underline{\kappa})$ is uncorrelated for all values of $\phi' \neq \phi$ and $\underline{\kappa}' \neq \underline{\kappa}$.

Therefore, in summary, under the assumption of <u>uncorrelated spreading</u> the four filter autocorrelation functions reduce as follows:

$$R_S(\tau,\tau',\underline{r}_o,\underline{r}_o';\ \phi,\phi',\underline{\kappa},\underline{\kappa}') = R_S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})\,\delta(\tau-\tau')\,\delta(\underline{r}_o-\underline{r}_o')\cdot \delta(\phi-\phi')\,\delta(\underline{\kappa}-\underline{\kappa}') \tag{2.2-16}$$

$$R_H(f,f',\underline{\nu},\underline{\nu}';t,t',\underline{r},\underline{r}') = R_H(\Delta f,\Delta\underline{\nu};\ \Delta t,\Delta\underline{r}) \tag{2.2-20}$$

$$R_h(\tau,\tau',\underline{r}_o,\underline{r}_o';\ t,t',\underline{r},\underline{r}') = R_h(\tau,\underline{r}_o;\ \Delta t,\Delta\underline{r})\,\delta(\tau-\tau')\,\delta(\underline{r}_o-\underline{r}_o') \tag{2.2-22}$$

and

$$R_B(f,f',\underline{\nu},\underline{\nu}';\ \phi,\phi',\underline{\kappa},\underline{\kappa}') = R_B(\Delta f,\Delta\underline{\nu};\phi,\underline{\kappa})\,\delta(\phi-\phi')\,\delta(\underline{\kappa}-\underline{\kappa}') \tag{2.2-24}$$

where $R_S(\tau,\underline{r}_o;\ \phi,\underline{\kappa})$ is the scattering function.

2.2.3 The scattering function and its Fourier transforms

By inspecting Eq. (2.2-23), we can write that

$$R_h(\tau,\underline{r}_o\ ;\ \Delta t,\Delta\underline{r}) = F_\phi^{-1}F_{\underline{\kappa}}^{-1}\{R_S(\tau,\underline{r}_o\ ;\ \phi,\underline{\kappa})\} \tag{2.2-26}$$

and

$$R_S(\tau,\underline{r}_o\ ;\ \phi,\underline{\kappa}) = F_{\Delta t}F_{\Delta\underline{r}}\{R_h(\tau,\underline{r}_o\ ;\ \Delta t,\Delta\underline{r})\} \tag{2.2-27}$$

or

$$R_S(\tau,\underline{r}_o\ ;\ \phi,\underline{\kappa}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_h(\tau,\underline{r}_o\ ;\ \Delta t,\ \Delta\underline{r})\exp(-j2\pi\phi\Delta t)\cdot \exp(+j2\pi\underline{\kappa}\cdot\Delta\underline{r})\,d\Delta t\,d\Delta\underline{r}. \tag{2.2-28}$$

Also, by inspecting Eq. (2.2-25), we can write that

$$R_B(\Delta f,\Delta\underline{\nu}\ ;\ \phi,\underline{\kappa}) = F_\tau F_{\underline{r}_o}\{R_S(\tau,\underline{r}_o\ ;\ \phi,\underline{\kappa})\} \tag{2.2-29}$$

and

$$R_S(\tau,\underline{r}_o\ ;\ \phi,\underline{\kappa}) = F_{\Delta f}^{-1}F_{\Delta\underline{\nu}}^{-1}\{R_B(\Delta f,\Delta\underline{\nu}\ ;\ \phi,\underline{\kappa})\} \tag{2.2-30}$$

or

$$R_S(\tau,\underline{r}_o\ ;\ \phi,\underline{\kappa}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_B(\Delta f,\Delta\underline{\nu}\ ;\ \phi,\underline{\kappa})\exp(+j2\pi\Delta f\tau)\cdot \exp(-j2\pi\Delta\underline{\nu}\cdot\underline{r}_o)\,d\Delta f\,d\Delta\underline{\nu}. \tag{2.2-31}$$

Additional transform pairs can be obtained as follows. With the use of Eq. (2.2-23), Eq. (2.2-21) can be expressed as

$$R_H(\Delta f,\Delta \underline{\nu}\ ;\ \Delta t,\Delta \underline{r}) = F_\tau F_{\underline{r}_o}\{R_h(\tau,\underline{r}_o\ ;\ \Delta t,\Delta \underline{r})\} \tag{2.2-32}$$

or

$$R_H(\Delta f,\Delta \underline{\nu}\ ;\ \Delta t,\Delta \underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_h(\tau,\underline{r}_o\ ;\ \Delta t,\Delta \underline{r})\exp(-j2\pi\Delta f\tau)\cdot \exp(+j2\pi\Delta \underline{\nu}\cdot\underline{r}_o)d\tau d\underline{r}_o \tag{2.2-33}$$

and

$$R_h(\tau,\underline{r}_o\ ;\ \Delta t,\Delta \underline{r}) = F^{-1}_{\Delta f}F^{-1}_{\Delta \underline{\nu}}\{R_H(\Delta f,\Delta \underline{\nu}\ ;\ \Delta t,\Delta \underline{r})\} \tag{2.2-34}$$

or

$$R_h(\tau,\underline{r}_o\ ;\ \Delta t,\ \Delta \underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_H(\Delta f,\Delta \underline{\nu}\ ;\ \Delta t,\Delta \underline{r})\exp(+j2\pi\Delta f\tau)\cdot \exp(-j2\pi\Delta \underline{\nu}\cdot\underline{r}_o)d\Delta f d\Delta \underline{\nu}. \tag{2.2-35}$$

Next, if Eq. (2.2-28) is substituted into Eq. (2.2-25), then

$$R_B(\Delta f,\Delta \underline{\nu}\ ;\ \phi,\underline{\kappa}) = F_\tau F_{\underline{r}_o} F_{\Delta t} F_{\Delta \underline{r}}\{R_h(\tau,\underline{r}_o\ ;\ \Delta t,\Delta \underline{r})\} \tag{2.2-36}$$

or

$$R_B(\Delta f,\Delta\underline{\nu}\ ;\ \phi,\underline{\kappa}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_h(\tau,\underline{r}_o\ ;\ \Delta t,\Delta\underline{r})\exp(-j2\pi\Delta f\tau)\cdot \exp(+j2\pi\Delta\underline{\nu}\cdot\underline{r}_o)\exp(-j2\pi\phi\Delta t)\cdot \exp(+j2\pi\underline{\kappa}\cdot\Delta\underline{r})\,d\tau\, d\underline{r}_o\, d\Delta t\, d\Delta\underline{r} \tag{2.2-37}$$

and

$$R_h(\tau,\underline{r}_o\ ;\ \Delta t,\Delta\underline{r}) = F^{-1}_{\Delta f}F^{-1}_{\Delta\underline{\nu}}F^{-1}_{\phi}F^{-1}_{\underline{\kappa}}\{R_B(\Delta f,\Delta\underline{\nu}\ ;\ \phi,\underline{\kappa})\} \tag{2.2-38}$$

or

$$R_h(\tau,\underline{r}_o\ ;\ \Delta t,\Delta\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_B(\Delta f,\Delta\underline{\nu}\ ;\ \phi,\underline{\kappa})\exp(+j2\pi\Delta f\tau)\cdot \exp(-j2\pi\Delta\underline{\nu}\cdot\underline{r}_o)\exp(+j2\pi\phi\Delta t)\cdot \exp(-j2\pi\underline{\kappa}\cdot\Delta\underline{r})\,d\Delta f\, d\Delta\underline{\nu}\, d\phi\, d\underline{\kappa}. \tag{2.2-39}$$

Finally, with the use of Eq. (2.2-33), Eq. (2.2-37) can be expressed as

$$R_B(\Delta f,\Delta\underline{\nu}\ ;\ \phi,\underline{\kappa}) = F_{\Delta t}F_{\Delta\underline{r}}\{R_H(\Delta f,\Delta\underline{\nu}\ ;\ \Delta t,\Delta\underline{r})\} \tag{2.2-40}$$

or

$$R_B(\Delta f,\Delta\underline{\nu}\ ;\ \phi,\underline{\kappa}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_H(\Delta f,\Delta\underline{\nu}\ ;\ \Delta t,\Delta\underline{r})\exp(-j2\pi\phi\Delta t)\cdot \exp(+j2\pi\underline{\kappa}\cdot\Delta\underline{r})\,d\Delta t\, d\Delta\underline{r} \tag{2.2-41}$$

and

$$R_H(\Delta f,\Delta \underline{\nu}\ ;\ \Delta t,\Delta \underline{r}) = F_{\phi}^{-1}F_{\underline{\kappa}}^{-1}\{R_B(\Delta f,\Delta \underline{\nu}\ ;\ \phi,\underline{\kappa})\} \tag{2.2-42}$$

or

$$R_H(\Delta f,\Delta \underline{\nu}\ ;\ \Delta t,\Delta \underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_B(\Delta f,\Delta \underline{\nu}\ ;\ \phi,\underline{\kappa})\exp(+j2\pi\phi\Delta t)\cdot \exp(-j2\pi\underline{\kappa}\cdot\Delta\underline{r})\,d\phi\, d\underline{\kappa}. \tag{2.2-43}$$

The various Fourier transform pairs are summarized in Fig. 6. The scattering function, or any of its Fourier transforms, is a complete second order statistical description of a WSSUS communication channel.

Another very important Fourier transform relationship can be obtained by substituting Eq. (2.2-35) into Eq. (2.2-28) which yields

$$R_S(\tau,\underline{r}_o\ ;\ \phi,\underline{\kappa}) = F_{\Delta t}F_{\Delta \underline{r}}F_{\Delta f}^{-1}F_{\Delta \underline{\nu}}^{-1}\{R_H(\Delta f,\Delta \underline{\nu}\ ;\ \Delta t,\Delta \underline{r})\} \tag{2.2-44}$$

or

$$R_S(\tau,\underline{r}_o\ ;\ \phi,\underline{\kappa}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_H(\Delta f,\Delta \underline{\nu}\ ;\ \Delta t,\Delta \underline{r})\exp(-j2\pi\phi\Delta t)\cdot \exp(+j2\pi\underline{\kappa}\cdot\Delta\underline{r})\exp(+j2\pi\Delta f\tau)\cdot \exp(-j2\pi\Delta\underline{\nu}\cdot\underline{r}_o)\,d\Delta t\, d\Delta\underline{r}\, d\Delta f\, d\Delta\underline{\nu} \tag{2.2-45}$$

and

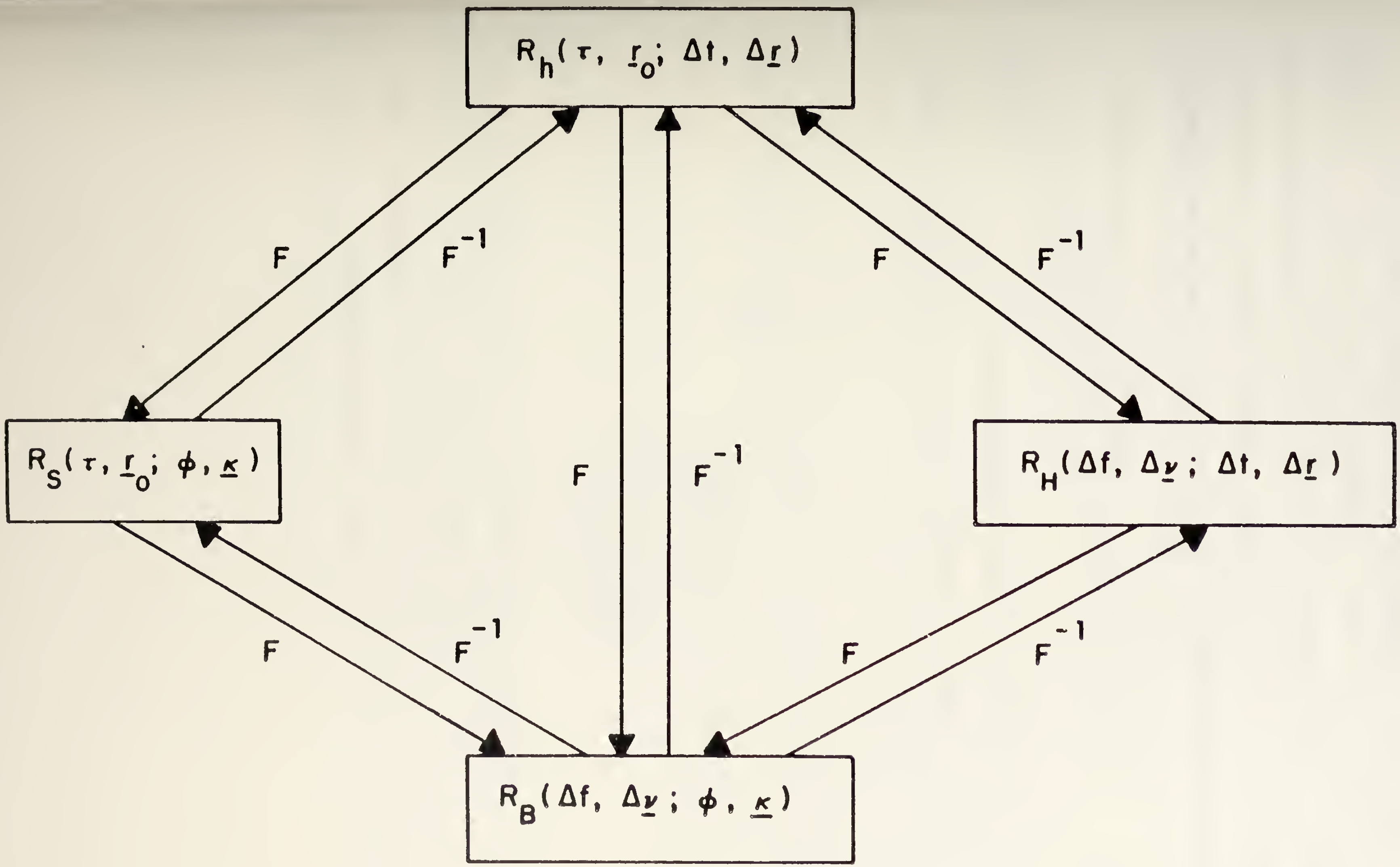

Fig. 6. The scattering function $R_S(\tau, \underline{r}_o ; \phi, \underline{\kappa})$ and its Fourier transforms.

$$R_H(\Delta f, \Delta \underline{\nu} \; ; \; \Delta t, \Delta \underline{r}) = F_{\phi}^{-1} F_{\underline{\kappa}}^{-1} F_{\tau} F_{\underline{r}_o} \{R_S(\tau, \underline{r}_o \; ; \; \phi, \underline{\kappa})\} \tag{2.2-46}$$

since

$$R_H(\Delta f, \Delta \underline{\nu} \; ; \; \Delta t, \Delta \underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_S(\tau, \underline{r}_o \; ; \; \phi, \underline{\kappa}) \exp(+j2\pi\phi\Delta t) \cdot \exp(-j2\pi\underline{\kappa}\cdot\Delta\underline{r}) \exp(-j2\pi\Delta f\tau) \cdot \exp(+j2\pi\Delta\underline{\nu}\cdot\underline{r}_o)\, d\phi\, d\underline{\kappa}\, d\tau\, d\underline{r}_o . \tag{2.2-21}$$

2.2.4 Input-output relations

We will now proceed to derive an expression for the autocorrelation function of the output from a linear, time-varying, space-varying, random filter. The output autocorrelation function $R_y(t,t',\underline{r},\underline{r}')$ is defined as follows:

$$R_y(t,t',\underline{r},\underline{r}') \triangleq E\{y(t',\underline{r})y^*(t\ \underline{r}')\}. \qquad (2.2\text{-}47)$$

If Eq. (2.1-30) is substituted into Eq. (2.2-47), then the output correlation function in time and space is given by

$$R_y(t,t',\underline{r},\underline{r}') = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x(t-\tau,\ \underline{r}-\underline{r}_o)x^*(t'-\tau',\ \underline{r}'-\underline{r}_o') \cdot$$
$$\exp[+j2\pi(\phi t-\phi' t')]\exp[-j2\pi(\underline{\kappa}\cdot\underline{r} - \underline{\kappa}'\cdot\underline{r}')] \cdot$$
$$R_S(\tau,\tau',\underline{r}_o,\underline{r}_o'\ ;\ \phi,\phi',\underline{\kappa},\underline{\kappa}')d\phi d\underline{\kappa} d\tau d\underline{r}_o \cdot$$
$$d\phi' d\underline{\kappa}' d\tau' d\underline{r}_o' \ . \qquad (2.2\text{-}48)$$

If the random filter is a WSSUS (wide-sense stationary uncorrelated spreading) communication channel, then the autocorrelation function of the spreading function is given by Eq. (2.2-16). Substituting Eq. (2.2-16) into Eq. (2.2-48) yields

$$R_y(t,t',\underline{r},\underline{r}') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\tau,\underline{r}-\underline{r}_o)x^*(t'-\tau,\underline{r}'-\underline{r}_o)\cdot$$
$$\exp(+j2\pi\phi\Delta t)\exp(-j2\pi\underline{\kappa}\cdot\Delta\underline{r})\cdot$$
$$R_S(\tau,\underline{r}_o\ ;\ \phi,\underline{\kappa})d\phi d\underline{\kappa}d\tau d\underline{r}_o. \qquad (2.2\text{-}49)$$

The mean squared value of $y(t,\underline{r})$ is given by

$$E\{|y(t,\underline{r})|^2\} = R_y(t,t,\underline{r},\underline{r})$$

where, from Eq. (2.2-48), we have in general that

$$R_y(t,t,\underline{r},\underline{r}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x(t-\tau,\underline{r}-\underline{r}_o)x^*(t-\tau',\underline{r}-\underline{r}_o')\cdot$$
$$\exp(+j2\pi\Delta\phi t)\exp(-j2\pi\Delta\underline{\kappa}\cdot\underline{r})\cdot$$
$$R_S(\tau,\tau',\underline{r}_o,\underline{r}_o'\ ;\ \phi,\phi',\underline{\kappa},\underline{\kappa}')d\phi d\underline{\kappa}d\tau d\underline{r}_o\cdot$$
$$d\phi' d\underline{\kappa}' d\tau' d\underline{r}_o' \qquad (2.2\text{-}50)$$

where $\Delta\phi = \phi-\phi'$ and $\Delta\underline{\kappa} = \underline{\kappa}-\underline{\kappa}'$. In the case of a WSSUS communication channel, we have from Eq. (2.2-49) that

$$R_y(t,t,\underline{r},\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x(t-\tau,\ \underline{r}-\underline{r}_o)|^2 R_S(\tau,\underline{r}_o\ ;\ \phi,\underline{\kappa})\cdot d\phi d\underline{\kappa} d\tau d\underline{r}_o. \tag{2.2-51}$$

If we define the energy of the output signal E_y as

$$E_y \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |y(t,\underline{r})|^2 dt d\underline{r}, \tag{2.2-52}$$

then the average output energy $\bar{E}_y$ can be expressed as

$$\bar{E}_y \triangleq E\{E_y\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{|y(t,\underline{r})|^2\} dt d\underline{r} \tag{2.2-53}$$

or

$$\bar{E}_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_y(t,t,\underline{r},\underline{r}) dt d\underline{r}. \tag{2.2-54}$$

If Eq. (2.2-51) is substituted into Eq.(2.2-54), then

$$\bar{E}_y/E_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_S(\tau,\underline{r}_o\ ;\ \phi,\underline{\kappa}) d\phi d\underline{\kappa} d\tau d\underline{r}_o \tag{2.2-55}$$

where

$$E_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x(t,\underline{r})|^2 dt d\underline{r} \qquad (2.2\text{-}56)$$

is the energy of the input signal. Equation (2.2-55) indicates that the ratio of the output (received) average energy to the input (transmitted) energy for a WSSUS communication channel can be obtained by integrating the scattering function of the channel. Also note that the average output energy is not a function of the input signal's shape.

Alternate expressions for the output autocorrelation function can be obtained from Eq. (2.1-25). If Eq. (2.1-25) is substituted into Eq. (2.2-47), then

$$\begin{aligned} R_y(t,t',\underline{r},\underline{r}') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & X(f,\underline{\nu})X^*(f',\underline{\nu}')\cdot \\ & R_H(f,f',\underline{\nu},\underline{\nu}' \ ; \ t,t',\underline{r},\underline{r}')\cdot \\ & \exp[+j2\pi(ft-f't')]\exp[-j2\pi(\underline{\nu}\cdot\underline{r}-\underline{\nu}'\cdot\underline{r}')] \\ & df d\underline{\nu} df' d\underline{\nu}'. \qquad (2.2\text{-}57) \end{aligned}$$

If the random filter is a WSSUS communication channel, then the autocorrelation function of the transfer function is given by Eq. (2.2-20). Substituting Eq. (2.2-20) into Eq. (2.2-57) yields

$$R_y(t,t',\underline{r},\underline{r}') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f,\underline{\nu})X^*(f',\underline{\nu}')R_H(\Delta f,\Delta\underline{\nu}\ ;\ \Delta t,\Delta\underline{r})\cdot$$
$$\exp[+j2\pi(ft - f't')]\exp[-j2\pi(\underline{\nu}\cdot\underline{r} - \underline{\nu}'\cdot\underline{r}')]\cdot$$
$$df d\underline{\nu} df' d\underline{\nu}'. \qquad (2.2\text{-}58)$$

A relationship between the input and output power spectral densities will be obtained next.

Let us assume that the input signal $x(t,\underline{r})$ is a zero-mean, wide-sense stationary (in time and space), random process which is uncorrelated with the transfer function. Under these assumptions, the output autocorrelation function given by Eq. (2.2-57) becomes

$$R_y(t,t',\underline{r},\underline{r}') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{X(f,\underline{\nu})X^*(f',\underline{\nu}')\}\cdot$$
$$R_H(f,f',\underline{\nu},\underline{\nu}'\ ;\ t,t',\underline{r},\underline{r}')\cdot$$
$$\exp[+j2\pi(ft-f't')]\exp[-j2\pi(\underline{\nu}\cdot\underline{r}-\underline{\nu}'\cdot\underline{r}')]\cdot$$
$$df d\underline{\nu} df' d\underline{\nu}' . \qquad (2.2\text{-}59)$$

Since $x(t,\underline{r})$ was assumed to be wide-sense stationary in both time and space, then it can be shown that

$$E\{X(f,\underline{\nu})X^*(f',\underline{\nu}')\} = S_x(f,\underline{\nu})\,\delta(f-f')\,\delta(\underline{\nu}-\underline{\nu}') \qquad (2.2\text{-}60)$$

where

$$X(f,\underline{\nu}) = F_t F_{\underline{r}}\{x(t,\underline{r})\},$$

$$S_x(f,\underline{\nu}) \triangleq F_{\Delta t}F_{\Delta \underline{r}}\{R_x(\Delta t,\Delta\underline{r})\} \tag{2.2-61}$$

is the power spectral density of the input, and

$$R_x(\Delta t,\Delta\underline{r}) = E\{x(t,\underline{r})x^*(t',\underline{r}')\}$$

where $\Delta t = t-t'$ and $\Delta\underline{r} = \underline{r}-\underline{r}'$. If Eq. (2.2-60) is substituted into Eq. (2.2-59), then

$$R_y(t,t',\underline{r},\underline{r}') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(f,\underline{\nu})R_H(f,f,\underline{\nu},\underline{\nu}\ ;\ t,t',\underline{r},\underline{r}')\exp(+j2\pi f\Delta t)\cdot \exp(-j2\pi\underline{\nu}\cdot\Delta\underline{r})\,df\,d\underline{\nu}, \tag{2.2-62}$$

and if it is further assumed that the transfer function is wide-sense stationary in both time and space, then Eq. (2.2-62) reduces to

$$R_y(\Delta t,\Delta\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(f,\underline{\nu})R_H(f,f,\underline{\nu},\underline{\nu}\ ;\ \Delta t,\Delta\underline{r})\cdot \exp(+j2\pi f\Delta t)\exp(-j2\pi\underline{\nu}\cdot\Delta\underline{r})\,df\,d\underline{\nu} \tag{2.2-63}$$

which implies that the output $y(t,\underline{r})$ is also wide-sense stationary in both time and space. If we define the output power spectral density $S_y(\eta,\underline{\beta})$ as

$$S_y(\eta,\underline{\beta}) \triangleq F_{\Delta t}F_{\Delta\underline{r}}\{R_y(\Delta t,\Delta\underline{r})\}, \tag{2.2-64}$$

then substituting Eq. (2.2-63) into the R.H.S. of Eq. (2.2-64) yields

$$S_y(\eta,\underline{\beta}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(f,\underline{\nu})R_B(f,f,\underline{\nu},\underline{\nu}\ ;\ \eta-f,\ \underline{\beta}-\underline{\nu})dfd\underline{\nu} \tag{2.2-65}$$

where, from Eq. (2.2-41),

$$R_B(f,f,\underline{\nu},\underline{\nu}\ ;\ \eta-f,\underline{\beta}-\underline{\nu}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_H(f,f,\underline{\nu},\underline{\nu}\ ;\ \Delta t,\Delta\underline{r})\exp[-j2\pi(\eta-f)\Delta t]\cdot \exp[+j2\pi(\underline{\beta}-\underline{\nu})\cdot\Delta\underline{r}]d\Delta t d\Delta\underline{r}. \tag{2.2-66}$$

Note that Eq. (2.2-65) is in the form of a multidimensional convolution integral analogous to Eq. (2.1-46) for the deterministic case. The convolution process accounts for the frequency spreading (frequency in HZ. and spatial frequencies in cycles/m.) of the input power spectral density.

<u>Example 2.2-1</u>

Using Eqs. (2.2-65) and (2.2-66), let us compute the output power spectral density for the case when the transfer function is time-<u>in</u>variant, space-<u>in</u>variant, and <u>deterministic</u>.

First note that

$$R_H(f,f,\underline{\nu},\underline{\nu}\ ;\ \Delta t,\Delta\underline{r}) = E\{H(f,\underline{\nu}\ ;\ t,\underline{r})H^*(f,\underline{\nu}\ ;\ t',\underline{r}')\}.$$

If $H(f,\underline{\nu}\ ;\ t,\underline{r})$ is both time-invariant and space-invariant, then [see Eq. (2.1-16)]

$$H(f,\underline{\nu}\ ;\ t,\underline{r}) = H(f,\underline{\nu}),$$

and if the transfer function is also deterministic, then

$$\begin{aligned} R_H(f,f,\underline{\nu},\underline{\nu}\ ;\ \Delta t,\Delta\underline{r}) &= E\{|H(f,\underline{\nu})|^2\} \\ &= |H(f,\underline{\nu})|^2. \end{aligned}$$

Therefore, Eq. (2.2-66) becomes

$$\begin{aligned} R_B(f,f,\underline{\nu},\underline{\nu}\ ;\ \eta-f,\underline{\beta}-\underline{\nu}) &= |H(f,\underline{\nu})|^2 \int_{-\infty}^{\infty} \exp[-j2\pi(\eta-f)\Delta t]d\Delta t \cdot \\ &\quad \int_{-\infty}^{\infty} \exp[+j2\pi(\underline{\beta}-\underline{\nu})\cdot\Delta\underline{r}]d\Delta\underline{r} \\ &= |H(f,\underline{\nu})|^2\delta(\eta-f)\delta(\underline{\beta}-\underline{\nu}), \end{aligned}$$

and substituting this expression into Eq. (2.2-65) yields the desired result

$$S_y(\eta,\underline{\beta}) = |H(\eta,\underline{\beta})|^2 S_x(\eta,\underline{\beta})\ .$$

By replacing η with f and $\underline{\beta}$ with $\underline{\nu}$, we finally obtain [compare with Eq. (2.1-48)]

$$S_y(f,\underline{\nu}) = |H(f,\underline{\nu})|^2 S_x(f,\underline{\nu}) \ .$$

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III. COUPLING EQUATIONS

We will now present those equations which couple the transmitted and received electrical signals to the transfer function of the ocean medium via the transmit and receive far-field directivity functions. Referring to Fig. 2, the frequency and angular spectra of the input acoustic signal to the medium and the output electrical signal from the receive aperture (array) are given, respectively, by the following equations [2]:

$$X_M(f,\underline{\nu}) = \int_{-\infty}^{\infty} X(f,\underline{\alpha})D_T(f,\underline{\nu}-\underline{\alpha})d\underline{\alpha} \tag{3-1}$$

and

$$Y(\eta,\underline{\gamma}) = \int_{-\infty}^{\infty} Y_M(\eta,\underline{\beta})D_R(\eta,\underline{\gamma}-\underline{\beta})d\underline{\beta} \tag{3-2}$$

where the frequency and angular spectrum of the output acoustic signal from the medium is given by [see Eq. (2.1-46)]

$$Y_M(\eta,\underline{\beta}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_M(f,\underline{\nu})B_M(f,\underline{\nu}\ ;\ \eta-f,\underline{\beta}-\underline{\nu})dfd\underline{\nu} \tag{3-3}$$

where [see Eq. (2.1-32)]

$$B_M(f,\underline{\nu}\ ;\ \phi,\underline{\kappa}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_M(f,\underline{\nu}\ ;\ t,\underline{r})\exp(-j2\pi\phi t)\cdot$$

$$\exp(+j2\pi\underline{\kappa}\cdot\underline{r})\,dt\,d\underline{r} \qquad (3\text{-}4)$$

is the bi-frequency function, and $H_M(f,\underline{\nu}\ ;\ t,\underline{r})$ is the time-variant, space-variant, transfer function of the ocean medium. Equations (3-1) thru (3-4) are the basic <u>coupling equations</u>. However, w.r.t. Eq. (3-1), we have the following additional relationships:

$$X_M(f,\underline{\nu}) = F_t F_{\underline{r}}\{x_M(t,\underline{r})\} = \int_{-\infty}^{\infty}\int_V x_M(t,\underline{r})\exp(+j2\pi\underline{\nu}\cdot\underline{r})\,dV\cdot$$

$$\exp(-j2\pi ft)\,dt \qquad (3\text{-}5)$$

where [2]

$$x_M(t,\underline{r}) = \int_{-\infty}^{\infty} X(f,\underline{r})A_T(f,\underline{r})\exp(+j2\pi ft)\,df, \qquad (3\text{-}6)$$

$$X(f,\underline{\alpha}) = F_t F_{\underline{r}}\{x(t,\underline{r})\} = \int_{-\infty}^{\infty}\int_V x(t,\underline{r})\exp(+j2\pi\underline{\alpha}\cdot\underline{r})\,dV\cdot$$

$$\exp(-j2\pi ft)\,dt \qquad (3\text{-}7)$$

is the frequency and angular spectrum of the transmitted electrical signal, and

$$D_T(f,\underline{\alpha}) = F_{\underline{r}}\{A_T(f,\underline{r})\} = \int_V A_T(f,\underline{r})\exp(+j2\pi\underline{\alpha}\cdot\underline{r})dV \tag{3-8}$$

is the far-field directivity function (beam pattern) of the complex transmit aperture $A_T(f,\underline{r})$. Also, w.r.t. Eq. (3-6),

$$X(f,\underline{r}) = F_t\{x(t,\underline{r})\} = \int_{-\infty}^{\infty} x(t,\underline{r})\exp(-j2\pi ft)dt \tag{3-9}$$

and

$$A_T(f,\underline{r}) = F_{\underline{\alpha}}^{-1}\{D_T(f,\underline{\alpha})\} = \int_{-\infty}^{\infty} D_T(f,\underline{\alpha})\exp(-j2\pi\underline{\alpha}\cdot\underline{r})d\underline{\alpha} \ . \tag{3-10}$$

Similarly, w.r.t. Eq. (3-2), we have the following additional relationships:

$$Y(\eta,\underline{\gamma}) = F_t F_{\underline{r}}\{y(t,\underline{r})\} = \int_{-\infty}^{\infty}\int_V y(t,\underline{r})\exp(+j2\pi\underline{\gamma}\cdot\underline{r})dV\cdot \exp(-j2\pi\eta t)dt \tag{3-11}$$

where [2]

$$y(t,\underline{r}) = \int_{-\infty}^{\infty} Y_M(\eta,\underline{r}) A_R(\eta,\underline{r}) \exp(+j2\pi\eta t)\, d\eta, \qquad (3\text{-}12)$$

the frequency and angular spectrum of the output acoustic signal from the medium is given by

$$Y_M(\eta,\underline{\beta}) = F_t F_{\underline{r}}\{y_M(t,\underline{r})\} = \int_{-\infty}^{\infty}\int_V y_M(t,\underline{r}) \exp(+j2\pi\underline{\beta}\cdot\underline{r})\, dV \cdot \exp(-j2\pi\eta t)\, dt \qquad (3\text{-}13)$$

where [see Eq.(2.1-25)]

$$y_M(t,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_M(f,\underline{\nu}) H_M(f,\underline{\nu}\ ;\ t,\underline{r}) \exp(+j2\pi f t) \cdot \exp(-j2\pi\underline{\nu}\cdot\underline{r})\, df\, d\underline{\nu}\ , \qquad (3\text{-}14)$$

and

$$D_R(\eta,\underline{\beta}) = F_{\underline{r}}\{A_R(\eta,\underline{r})\} = \int_V A_R(\eta,\underline{r}) \exp(+j2\pi\underline{\beta}\cdot\underline{r})\, dV \qquad (3\text{-}15)$$

is the far-field directivity function (beam pattern) of the complex receive aperture $A_R(\eta,\underline{r})$. Finally, w.r.t. Eq. (3-12),

$$Y_M(\eta,\underline{r}) = F_t\{y_M(t,\underline{r})\} = \int_{-\infty}^{\infty} y_M(t,\underline{r})\exp(-j2\pi\eta t)dt \tag{3-16}$$

and

$$A_R(\eta,\underline{r}) = F_{\underline{\beta}}^{-1}\{D_R(\eta,\underline{\beta})\} = \int_{-\infty}^{\infty} D_R(\eta,\underline{\beta})\exp(-j2\pi\underline{\beta}\cdot\underline{r})d\underline{\beta}. \tag{3-17}$$

Substituting Eq. (3-14) into Eq. (3-16) yields

$$Y_M(\eta,\underline{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_M(f,\underline{\nu})B_M(f,\underline{\nu}\ ;\ \eta-f,\underline{r})\exp(-j2\pi\underline{\nu}\cdot\underline{r})dfd\underline{\nu} \tag{3-18}$$

where

$$B_M(f,\underline{\nu}\ ;\ \phi,\underline{r}) = \int_{-\infty}^{\infty} H_M(f,\underline{\nu}\ ;\ t,\underline{r})\exp(-j2\pi\phi t)dt \tag{3-19}$$

which is not the same as the bi-frequency function [see Eq. (3-4)].

The components of the vectors $\underline{\alpha}$, $\underline{\nu}$, $\underline{\beta}$, and $\underline{\gamma}$ are spatial frequencies with units of cycles/m. and $\underline{r} = (x,y,z)$.

A moments reflection leads one to the conclusion that Eq. (3-12) is well suited for space-time signal processing applications. By inspecting Eqs. (3-18), (3-19), and (3-1), it can be seen that the output signal given by Eq. (3-12) can be expressed as a function of time and space in terms of the transmitted electrical signal, the transmit aperture, the transfer function of the ocean medium, and the receive aperture. Equation (3-2), however, represents the theoretical output "spectrum", albeit dependent upon the same system functions.

If the ocean medium is modelled as a random filter, then $y(t,\underline{r})$ given by Eq. (3-12) is a random process with autocorrelation function

$$R_y(t,t',\underline{r},\underline{r}') = E\{y(t,\underline{r})y^*(t',\underline{r}')\}. \qquad (3\text{-}20)$$

It is probably obvious by now that the weak link in the coupling equations is the ocean medium transfer function $H_M(f,\underline{\nu}\ ;\ t,\underline{r})$ and its corresponding autocorrelation function [see Eq. (2.2-2)]

$$R_{H_M}(f,f',\underline{\nu},\underline{\nu}'\ ;\ t,t',\underline{r},\underline{r}') = E\{H_M(f,\underline{\nu}\ ;\ t,\underline{r})H_M^*(f',\underline{\nu}'\ ;\ t',\underline{r}')\}. \qquad (3\text{-}21)$$

If one is not able to specify a realistic functional form for $H_M(f,\underline{\nu}\ ;\ t,\underline{r})$, then the equations presented in this section represent an exercise in the methods of linear system theory. This leads us to the main purpose of this paper which is to demonstrate the <u>derivation</u> of an ocean transfer function based upon the W.K.B. approximation. The derivation is discussed in Section 4.1.

Finally, note that the autocorrelation function given by Eq. (3-21) can be referred to as a generalized coherence function since it is a generalization of the two-frequency correlation function or two-frequency mutual coherence function based upon linear, time-variant filter theory as discussed by Ishimaru [51], for example. Equation (3-21) provides information concerning the amount of spreading in time-delay, space, frequency, and spatial frequencies that a transmitted signal will be expected to experience as it propagates in a random medium. The result of all this spreading is, of course, distortion in pulse shape.

IV. ANALYSIS

4.1 Transfer Function

We will now proceed to derive a transfer function which models the bistatic communication channel geometry shown in Fig. 1. The communication channel is regarded to be the ocean volume between the apertures so that surface and bottom scattering effects are not included. Both apertures are stationary (not in motion), and it is assumed that no discrete point scatterers (such as bubbles, fish, etc.) are in the volume between the apertures. No motion implies that the resulting transfer function will be time-invariant.

The propagation of small amplitude acoustic signals in the ocean from the transmit aperture to the receive aperture can be described by the following linear, inhomogeneous, scalar wave equation:

$$\nabla^2 \varphi(t,\underline{r}) - \frac{1}{c^2(\underline{r})} \frac{\partial^2}{\partial t^2} \varphi(t,\underline{r}) = x_M(t,\underline{r}) \tag{4.1-1}$$

where $\varphi(t,\underline{r})$ is the velocity potential at time t and position $\underline{r} = (x,y,z)$, $x_M(t,\underline{r})$ is the source distribution [see Fig. 2 and Eq. (3-6)], and $c(\underline{r})$ is the speed of sound in the ocean. Since the coupling equations discussed in Section III already allow for an arbitrary $x_M(t,\underline{r})$ with corresponding frequency and angular spectrum $X_M(f,\underline{\nu})$, we need only find the solution to the following Helmholtz wave equation:

$$\nabla^2 \varphi(\underline{r}) + k_o^2\, n^2(\underline{r})\varphi(\underline{r}) = 0 \qquad (4.1\text{-}2)$$

where

$$k_o = 2\pi f/c_o = 2\pi/\lambda_o \qquad (4.1\text{-}3)$$

is the constant, reference wavenumber,

$$n(\underline{r}) = c_o/c(\underline{r}) \qquad (4.1\text{-}4)$$

is the random index of refraction,

$$c_o = c(\underline{r}_o) = f\lambda_o \qquad (4.1\text{-}5)$$

is the constant, reference speed of sound at the source position $\underline{r}_o = (x_o, y_o, z_o)$, and

$$\varphi(t,\underline{r}) = \varphi(\underline{r})\exp(+j2\pi ft) \qquad (4.1\text{-}6)$$

is the time-harmonic solution of Eq. (4.1-1) when $x_M(t,\underline{r})$ is set equal to zero,and where $\varphi(\underline{r})$ is the solution of Eq. (4.1-2). Note that the wavenumber

$$k(\underline{r}) = 2\pi f/c(\underline{r}) \qquad (4.1\text{-}7)$$

can be expressed as

$$k(\underline{r}) = k_o n(\underline{r}). \qquad (4.1\text{-}8)$$

Therefore, $k(\underline{r}_o) = k_o$ since $n(\underline{r}_o) = 1$. The index of refraction is commonly written as [40, 42-45]

$$n(\underline{r}) = n_D(\underline{r}) + n_R(\underline{r}) \tag{4.1-9}$$

or

$$n(\underline{r}) = n_D(\underline{r}) + \sigma(\underline{r})n_{NR}(\underline{r}) \tag{4.1-10}$$

where $n_D(\underline{r})$ is the deterministic component and is usually close to unity in value, $n_R(\underline{r})$ is the random, zero mean component, $\sigma(\underline{r})$ is the standard deviation of $n_R(\underline{r})$, and

$$n_{NR}(\underline{r}) = n_R(\underline{r})/\sigma(\underline{r}) \tag{4.1-11}$$

is the normalized random component with zero mean and variance equal to unity. We shall work with Eq. (4.1-10) in this paper. Note that the average value of $n(\underline{r})$ is equal to $n_D(\underline{r})$.

Let us assume that the speed of sound is only a function of the depth y, i.e., $c(\underline{r}) = c(y)$, so that Eq. (4.1-2) reduces to

$$\nabla^2\varphi(x,y,z) + k_o^2\, n^2(y)\varphi(x,y,z) = 0 \tag{4.1-12}$$

where, from Eq. (4.1-10),

$$n(y) = n_D(y) + \sigma(y)n_{NR}(y). \tag{4.1-13}$$

By using the method of separation of variables and the W.K.B. approximation [46,47], an approximate solution of Eq. (4.1-12) is given by

$$\varphi(x,y,z) \approx \exp[-jk_X(x-x_o)]\ a_Y(y)\exp[+j\theta_Y(y)]\exp[-jk_Z(z-z_o)] \tag{4.1-14}$$

where

$$a_Y(y) = [k_Y(y)]^{-1/2} \tag{4.1-15}$$

$$\theta_Y(y) = \exp\left\{-j\int_{y_o}^{y} k_Y(\zeta)\,d\zeta\right\} \tag{4.1-16}$$

and

$$k_Y(y) = [k_o^2 n^2(y) - k_X^2 - k_Z^2]^{1/2} \tag{4.1-17}$$

where k_X, $k_Y(y)$, and k_Z are the components of the propagation vector

$$\underline{k}(y) = k_X\hat{x} + k_Y(y)\hat{y} + k_Z\hat{z}. \tag{4.1-18}$$

Note that k_X and k_Z are <u>constants</u> while $k_Y(y)$ is a function of the depth y. In addition, Eqs. (4.1-14) thru (4.1-16) allow for a general source location with position vector $\underline{r}_o = (x_o, y_o, z_o)$

The W.K.B. approximation given by Eqs. (4.1-15) and (4.1-16) is a valid solution for the depth dependence provided that [46,47,50] 1) the transmitted frequencies f are high, 2) the sound speed profile c(y) is slowly changing, and 3) the depth interval from y_o to y does not include any turning points. A turning point exists at $y = y_T$ if $k_Y(y_T) = 0$.

Since the reference propagation vector $\underline{k}_o$ can be expressed as

$$\underline{k}_o = k_X\hat{x} + k_Y\hat{y} + k_Z\hat{z} \ , \tag{4.1-19}$$

then

$$k_o^2 = |\underline{k}_o|^2 = k_X^2 + k_Y^2 + k_Z^2 \tag{4.1-20}$$

where

$$k_X = k_o u_o \tag{4.1-21}$$

$$k_Y = k_o v_o \tag{4.1-22}$$

and

$$k_Z = k_o w_o \tag{4.1-23}$$

where

$$u_o = \sin\theta_o \cos\psi_o \tag{4.1-24}$$

$$v_o = \sin\theta_o \sin\psi_o = \cos\beta_o \tag{4.1-25}$$

and

$$w_o = \cos\theta_o \tag{4.1-26}$$

are the direction cosines w.r.t. the positive X,Y, and Z axes, respectively, and (θ_o, ψ_o) are the vertical and azimuthal spherical angles measured w.r.t. the positive Z and X axes, respectively, representing the initial directions of wave propagation (see Fig. 7). From Eq. (4.1-20) we can write that

$$k_X^2 + k_Z^2 = k_o^2 - k_Y^2 \tag{4.1-27}$$

and substituting Eqs. (4.1-27) and (4.1-22) into Eq. (4.1-17) yields

$$k_Y(y) = k_Y \left\{ 1 + \frac{[n^2(y)-1]}{v_o^2} \right\}^{1/2} . \tag{4.1-28}$$

Note, that $k_Y(y_o) = k_Y$ since $n(y_o) = 1$. Therefore, the square root expression in Eq. (4.1-28) is responsible for changing the initial direction of propagation k_Y. If

$$|[n^2(y) - 1]/v_o^2| < 1 , \tag{4.1-29}$$

then the square root expression in Eq. (4.1-28) can be approximated by the first two terms in a binomial expansion yielding

$$k_Y(y) \approx k_Y + k_o^2[n^2(y) - 1]/(2k_Y) . \tag{4.1-30}$$

It will be shown later that the binomial expansion criterion given by Eq. (4.1-29) can be related to the critical angle of incidence, and hence, total reflection. The assumption concerning the absence of turning points will also be discussed later, but first let us return to the derivation of the transfer function.

Recall that we designated the vector $\underline{\nu}$ to represent the transmitted spatial frequencies as in $X_M(f,\underline{\nu})$ (see Fig. 2). If we let

$$\underline{\nu} = (f_X, f_Y, f_Z), \tag{4.1-31}$$

then

$$f_X = k_X/(2\pi) = u_o/\lambda_o \tag{4.1-32}$$

$$f_Y = k_Y/(2\pi) = v_o/\lambda_o \tag{4.1-33}$$

and

$$f_Z = k_Z/(2\pi) = w_o/\lambda_o \tag{4.1-34}$$

are the spatial frequencies in the X,Y, and Z directions, respectively, where $c_o = c(y_o) = f\lambda_o$. Therefore, if Eqs. (4.1-30) thru (4.1-34) are substituted into Eqs. (4.1-14) thru (4.1-16), then Eq. (4.1-6) becomes

$$\varphi(t,x,y,z) \approx \frac{\exp\left\{-j[k_o^2/(4\pi f_Y)]\int_{y_o}^{y}[n^2(\zeta)-1]d\zeta\right\}}{\left\{2\pi f_Y + k_o^2\,[n^2(y) - 1]\,/(4\pi f_Y)\right\}^{1/2}}\,.$$

$$\exp(+j2\pi ft)\exp[-j2\pi\underline{\nu}\cdot(\underline{r}-\underline{r}_o)]. \tag{4.1-35}$$

Since the input to the communication channel is the time-harmonic plane wave

$$\exp(+j2\pi ft)\exp[-j2\pi\underline{\nu}\cdot(\underline{r}-\underline{r}_o)],$$

and since Eq. (4.1-35) represents the output from the channel at any time t and position $\underline{r}$, then [see Eq. (2.1-21)]

$$H_M(f,\underline{\nu}\ ;\ t,\underline{r}) = H_M(f,\underline{\nu}\ ;\ \underline{r}) = H_M(f,f_Y\ ;\ y) \tag{4.1-36}$$

where the random, time-invariant, space-variant transfer function of the ocean medium is given by

$$H_M(f,f_Y\ ;\ y) = A_M(f,f_Y\ ;\ y)\exp[+j\theta_M(f,f_Y\ ;\ y)] \tag{4.1-37}$$

where

$$A_M(f,f_Y\ ;\ y) = \{2\pi f_Y + k_0^2[n^2(y)-1]/(4\pi f_Y)\}^{-1/2} \tag{4.1-38}$$

$$\theta_M(f,f_Y\ ;\ y) = -[k_0^2/(4\pi f_Y)]\int_{y_0}^{y}[n^2(\zeta)-1]d\zeta \tag{4.1-39}$$

$$k_0 = 2\pi f/c_0 \tag{4.1-3}$$

and from Eq. (4.1-13),

$$n^2(y) = n_D^2(y) + 2n_D(y)\sigma(y)n_{NR}(y) + \sigma^2(y)n_{NR}^2(y). \tag{4.1-40}$$

If the medium is "weakly irregular" or "weakly scattering", i.e., if $\sigma(y)$ is very small compared to unity so that

$$k_o \sigma(y) s_Y << 1 \qquad (4.1\text{-}41)$$

where s_Y is the scale size of the irregularities (i.e., the average distance over which the refractive index fluctuations remain correlated [48]), then terms involving $\sigma^2(y)$ can be neglected [42] and Eq. (4.1-40) reduces to

$$n^2(y) \approx n_D^2(y) + 2n_D(y)\sigma(y)n_{NR}(y). \qquad (4.1\text{-}42)$$

Therefore, by using Eq. (4.1-29) to simplify Eq. (4.1-38) and upon substituting Eq. (4.1-42) into Eq. (4.1-39), one obtains the following:

$$A_M(f,f_Y \; ; \; y) = (2\pi f_Y)^{-1/2} \qquad (4.1\text{-}43)$$

and

$$\Theta_M(f,f_Y \; ; \; y) = \Theta_{MD}(f,f_Y \; ; \; y) + \Theta_{MR}(f,f_Y \; ; \; y) \qquad (4.1\text{-}44)$$

where

$$\Theta_{MD}(f,f_Y \; ; \; y) = -[k_o^2/(4\pi f_Y)] \int_{y_o}^{y} [n_D^2(\zeta)-1]d\zeta \qquad (4.1\text{-}45)$$

is the deterministic or average component of the phase function, and

$$\Theta_{MR}(f,f_Y\ ;\ y) = -[k_o^2/(2\pi f_Y)] \int_{y_o}^{y} n_D(\zeta)\sigma(\zeta)n_{NR}(\zeta)d\zeta \tag{4.1-46}$$

is the random component.

The medium transfer function H_M given by Eq. (4.1-37) with amplitude and phase functions A_M and Θ_M specified by Eq. (4.1-43) and Eqs. (4.1-44) thru (4.1-46), respectively, indicate that for a weakly irregular medium, the major effect of the medium is to <u>angle modulate</u> the transmitted field. Furthermore, if $n_D(y) \approx 1$, then $\Theta_{MD}(f,f_Y\ ;\ y) \approx 0$ [see Eq. (4.1-45)] and, as a result, the angle modulation is due strictly to the random fluctuations $\sigma(y)n_{NR}(y)$ of the index of refraction [see Eq. (4.1-46)]. Note that the angle modulation process is often referred to as "scattering" [48]. Also note that the transfer function derived in this section can be written as the product of two functions, one deterministic and the other random, i.e.,

$$H_M = [A_M\exp(+j\Theta_{MD})][\exp(+j\Theta_{MR})]$$

which agrees with the assumed transfer function expressions of Laval and Labasque [36] and with the general practice of representing a field in a random medium as the product of a deterministic and a random function [40].

Let us now discuss the two important assumptions responsible for the derivation of the transfer function, namely, the aforementioned binomial expansion criterion given by Eq. (4.1-29) and the assumption regarding the absence of turning points.

Since

$$n(y) = c_o/c(y), \tag{4.1-47}$$

substituting Eqs. (4.1-25) and (4.1-47) into Eq. (4.1-29) yields

$$\beta_o < \cos^{-1}\left\{\frac{(|c^2(y) - c_o^2|)^{1/2}}{c(y)}\right\} \tag{4.1-48}$$

where β_o is the angle of incidence of the reference propagation vector $\underline{k}_o$ (see Fig. 7). Now recall that the critical angle of incidence β_c associated with a time-harmonic plane wave incident upon a plane boundary between two fluid media is given by [49] (see Fig. 8)

$$\sin\beta_c = c_1/c_2 \quad ; \quad c_2 > c_1 \tag{4.1-49}$$

or, equivalently,

$$\beta_c = \cos^{-1}\left\{\frac{(c_2^2 - c_1^2)^{1/2}}{c_2}\right\} ; \; c_2 > c_1 \tag{4.1-50}$$

where c_2 <u>must</u> be greater than c_1 for β_c to exist. When the angle of incidence $\beta \geq \beta_c$, there is <u>total reflection</u> and hence,

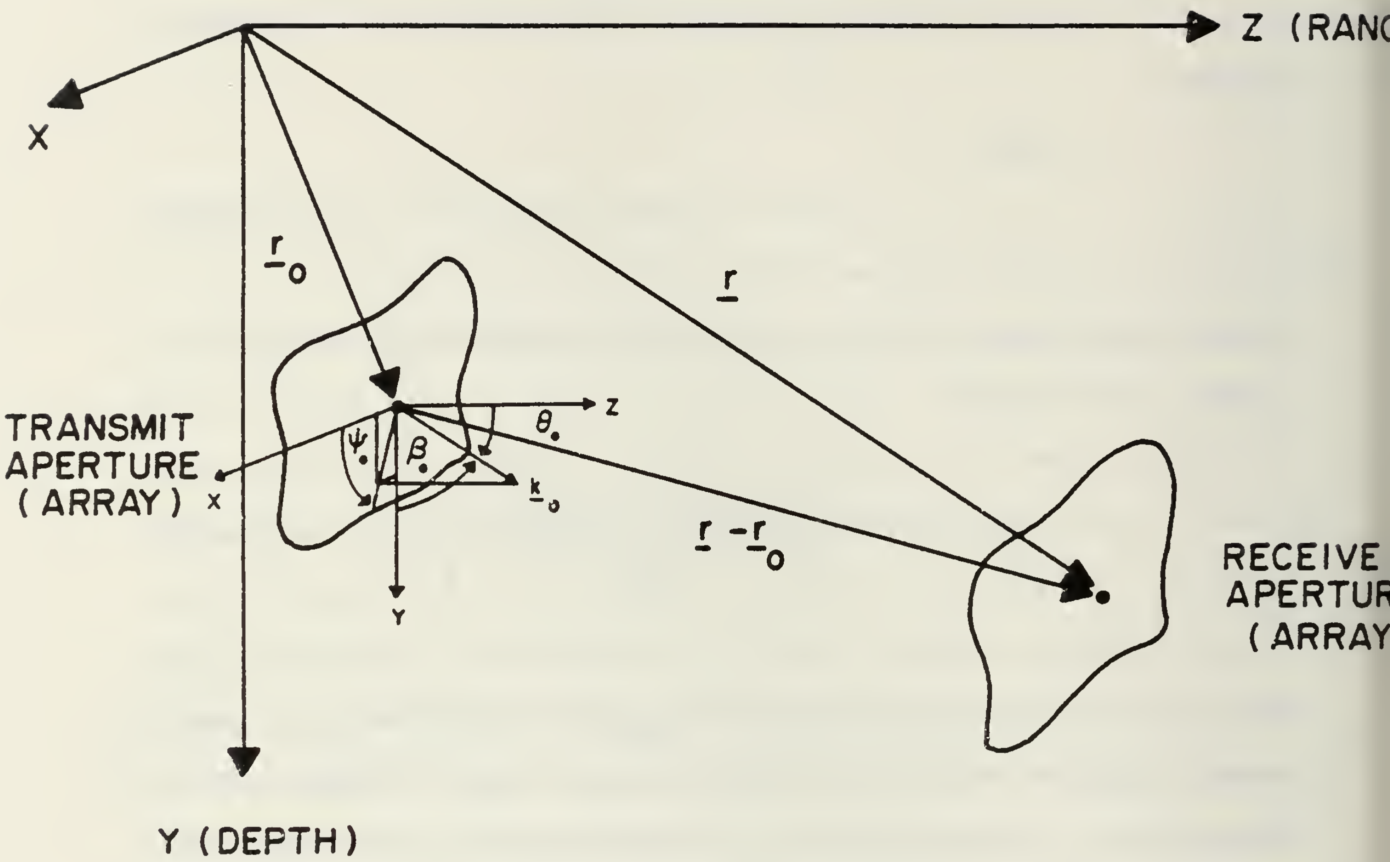

Fig. 7. The reference propagation vector $\underline{k}_o$ and associated angles.

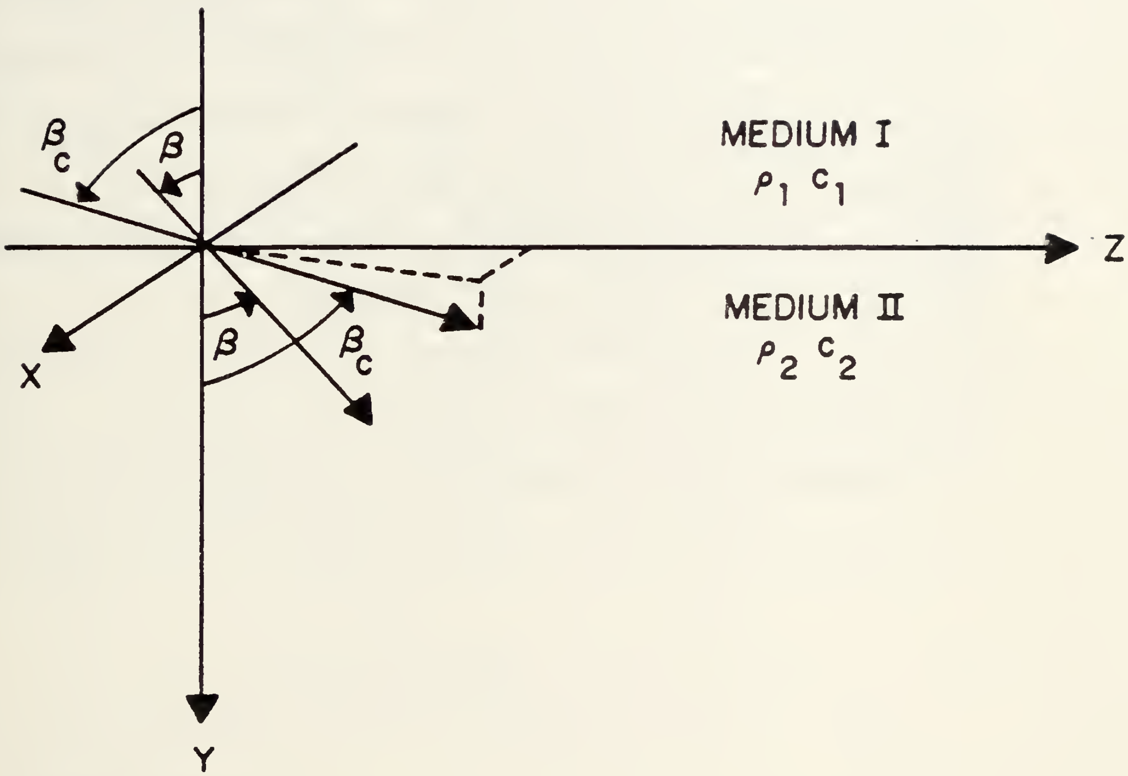

Fig. 8. Illustration of the angle of incidence β and the critical angle of incidence β_c.

no transmission of energy into medium II. The speed of sound c(y) generally increases with depth, for example, in the deep ocean. Also, c(y) increases above and below the SOFAR channel axis. Therefore, when $c(y) > c_o$, the absolute value sign in Eq. (4.1-48) can be removed, and by comparing Eq. (4.1-50) with the R.H.S. of Eq. (4.1-48), the binomial expansion criterion indicates that the initial angles of transmission β_o must be less than the "critical angle" in order to avoid total reflection, and thus, passing thru a turning point [47,50]. This is consistent with the fact that the W.K.B. approximation is invalid at a turning point [47,50].

4.2 Output Electrical Signal

Now that we have derived an ocean medium transfer function, let us demonstrate the use of the coupling equations presented in Section III by calculating the output electrical signal $y(t,\underline{r})$ from the receive aperture (array) as given by Eq. (3-12). As can be seen by inspecting the integrand of Eq. (3-12), we need expressions for both of the kernels $Y_M(\eta,\underline{r})$ and $A_R(\eta,\underline{r})$. Let us compute $Y_M(\eta,\underline{r})$ first [see Eq. (3-18)].

Assume that the transmit aperture depicted in Fig.1 is a planar array of $M \times N$ (odd) complex weighted point sources, centered at (x_o,y_o,z_o) and parallel to the XY plane. In addition, assume that the complex weights are separable. Since in most practical situations an identical input electrical signal is applied to all elements in the transmit array <u>before</u> the complex weights, i.e., since

$$x(t,\underline{r}) \equiv x(t),$$

then Eq. (3-7) reduces to

$$X(f,\underline{\alpha}) = X(f)\delta(\underline{\alpha}) \tag{4.2-1}$$

since $F_{\underline{r}}\{1\} = \delta(\underline{\alpha})$. Substituting Eq. (4.2-1) into Eq. (3-1) yields

$$X_M(f,\underline{\nu}) = X(f)D_T(f,\underline{\nu}) \tag{4.2-2}$$

where

$$D_T(f,\underline{\nu}) = \sum_{m=-(M-1)/2}^{(M-1)/2} \sum_{n=-(N-1)/2}^{(N-1)/2} c_m d_n \exp(+j2\pi f_X m d_X) \cdot$$

$$\exp(+j2\pi f_Y n d_Y) \exp(+j2\pi f_X x_0) \cdot$$

$$\exp(+j2\pi f_Y y_0) \exp(+j2\pi f_Z z_0)$$

$$(4.2\text{-}3)$$

is the far-field beam pattern of the transmit array, c_m and d_n are complex weights, d_X and d_Y are the interelement spacings in the X and Y directions, respectively, and the last three exponentials are phase factors which account for the array being centered at (x_0,y_0,z_0) instead of at the origin (0,0,0).

Since the transfer function derived in Section 4.1 is time-invariant, Eq. (3-19) becomes

$$B_M(f,\underline{\nu}\ ;\ \phi,\underline{r}) = H_M(f,\underline{\nu}\ ;\ \underline{r})\,\delta(\phi) \qquad (4.2\text{-}4)$$

and, as a result,

$$B_M(f,\underline{\nu}\ ;\ \eta-f,\underline{r}) = H_M(f,\underline{\nu}\ ;\ \underline{r})\,\delta(\eta-f). \qquad (4.2\text{-}5)$$

Therefore, substituting Eqs. (4.2-2) and (4.2-5) into Eq. (3-18) yields

$$Y_M(\eta,\underline{r}) = X(\eta)\int_{-\infty}^{\infty} D_T(\eta,\underline{\nu})H_M(\eta,\underline{\nu}\ ;\ \underline{r})\exp(-j2\pi\underline{\nu}\cdot\underline{r})d\underline{\nu}, \tag{4.2-6}$$

and upon substituting Eq. (4.2-6) into Eq. (3-12), we obtain

$$y(t,\underline{r}) = \int_{-\infty}^{\infty} X(f)\int_{-\infty}^{\infty} D_T(f,\underline{\nu})H_M(f,\underline{\nu}\ ;\ \underline{r})\exp(-j2\pi\underline{\nu}\cdot\underline{r})d\underline{\nu}\cdot A_R(f,\underline{r})\exp(+j2\pi ft)df \tag{4.2-7}$$

where η was replaced by f since there is no frequency (HZ) spreading in this example due to the time-invariant property of the transfer function. We need to specify the complex aperture function $A_R(f,\underline{r})$ next.

Assume that the receive aperture depicted in Fig. 1 is a planar array of M'x N' (odd) complex weighted point sources, centered at (x_R, y_R, z_R) and parallel to the XY plane. In addition, assume that the complex weights are separable. Therefore, the receive aperture function is given by

$$A_R(f,\underline{r}) = \sum_{i=-(M'-1)/2}^{(M'-1)/2}\ \sum_{q=-(N'-1)/2}^{(N'-1)/2} c_i' d_q' \delta(x-[x_R + i d_X'])\cdot\delta(y-[y_R + q d_Y'])\delta(z-z_R) \tag{4.2-8}$$

where c_i' and d_q' are complex weights and d_X' and d_Y' are the inter-element spacings in the X and Y directions, respectively.

Upon substituting Eqs. (4.1-31), (4.1-36), (4.2-3), and (4.2-8) into Eq. (4.2-7) and recalling that $\underline{r} = (x,y,z)$, one obtains

$$y(t,x,y,z) = \sum_{i=-(M'-1)/2}^{(M'-1)/2} \sum_{q=-(N'-1)/2}^{(N'-1)/2} c_i' d_q' \int_{-\infty}^{\infty} X(f) \cdot$$

$$\sum_{m=-(M-1)/2}^{(M-1)/2} c_m \int_{-\infty}^{\infty} \exp(-j2\pi f_X \Delta X_{im}) df_X \cdot$$

$$\sum_{n=-(N-1)/2}^{(N-1)/2} d_n \int_{-\infty}^{\infty} H_M(f, f_Y \; ; \; y_R + qd_Y') \exp(-j2\pi f_Y \Delta Y_{qn}) df_Y$$

$$\int_{-\infty}^{\infty} \exp(-j2\pi f_Z \Delta Z) df_Z \exp(+j2\pi ft) df \cdot$$

$$\delta(x-[x_R + id_X'])\,\delta(y-[y_R + qd_Y'])\,\delta(z-z_R) \qquad (4.2\text{-}9)$$

where

$$\Delta X_{im} = (x_R - x_o) + (id_X' - md_X) \qquad (4.2\text{-}10)$$

$$\Delta Y_{qn} = (y_R - y_o) + (qd_Y' - nd_Y) \qquad (4.2\text{-}11)$$

and

$$\Delta Z = z_R - z_o. \qquad (4.2\text{-}12)$$

If we now change variables from spatial frequencies to direction cosines by substituting Eqs. (4.1-32) thru (4.1-34) into Eq. (4.2-9), and treat the frequency variable f as a constant with respect to the spatial frequency integrations, then Eq. (4.2-9) becomes

$$y(t,x,y,z) = (2/c_o^3) \sum_{i=-(M'-1)/2}^{(M'-1)/2} \sum_{q=-(N'-1)/2}^{(N'-1)/2} c_i' d_q' \int_{-\infty}^{\infty} f^3 X(f) \cdot$$

$$\left\{ \sum_{m=-(M-1)/2}^{(M-1)/2} c_m \operatorname{sinc}(2f\Delta X_{im}/c_o) \right\} \cdot$$

$$\left\{ \sum_{n=-(N-1)/2}^{(N-1)/2} d_n \int_{a_q}^{+1} H_M(f, [fv_o/c_o]; y_R + qd_Y') \cdot \right.$$

$$\left. \exp(-j2\pi[fv_o/c_o]\Delta Y_{qn}) dv_c \right\} \cdot$$

$$\{b_q \exp(-j\pi b_q f\Delta Z/c_o) \operatorname{sinc}(b_q f\Delta Z/c_o)\} \cdot$$

$$\exp(+j2\pi ft) df\, \delta(x - [x_R + id_X']) \delta(y - [y_R + qd_Y']) \delta(z - z_R) \tag{4.2-13}$$

since in our example problem

$$-1 \le u_o \le +1 \tag{4.2-14}$$

$$a_q < v_o \le +1 \tag{4.2-15}$$

and

$$0 \le w_o < b_q \tag{4.2-16}$$

where

$$a_q = (|n_D^2(y_R + qd_Y') - 1|)^{1/2} \tag{4.2-17}$$

$$b_q = (1 - |n_D^2(y_R + qd_Y') - 1|)^{1/2} \tag{4.2-18}$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \tag{4.2-19}$$

and $n^2(\cdot)$ was replaced by $n_D^2(\cdot)$ in Eqs. (4.2-17) and (4.2-18). The transfer function H_M is given by Eqs. (4.1-37) and (4.1-43) thru (4.1-46) with the exception that the lower limit of integration y_o in Eqs. (4.1-45) and (4.1-46) must be replaced by $y_o + nd_Y$ which is the Y coordinate of a point source in the transmit planar array. Note that if $n_D(\cdot) < 1$, then

$$b_q = n_D(y_R + qd_Y'). \tag{4.2-20}$$

Equation (4.2-13) is the desired result, i.e., it represents the random output electrical signal from each element in the receive array in terms of the transmitted electrical signal, the transmit and receive arrays, and the random transfer function of the ocean medium. If $y(t,x,y,z)$ given by Eq. (4.2-13) is complex, then simply take the real part to obtain the real output electrical signal.

4.3 Coherence Function

Upon inspecting Eq. (4.2-13), it can be seen that the autocorrelation function of the output electrical signal will depend upon the autocorrelation function of the transfer function

$$R_{H_M}(f,f',f_Y,f'_Y \; ; \; y,y') = E\{H_M(f,f_Y \; ; \; y)H^*_M(f',f'_Y \; ; \; y')\} \tag{4.3-1}$$

which is also known as the coherence function. Substituting Eqs. (4.1-37), (4.1-39), and (4.1-43) into Eq. (4.3-1) yields

$$R_{H_M}(f,f',f_Y,f'_Y \; ; \; y,y') = \left[2\pi(f_Y f'_Y)^{1/2}\right]^{-1} E\left\{\exp(+j\,[KI(y)+K'I(y')])\right\} \tag{4.3-2}$$

where

$$K = -\,k_o^2/(4\pi f_Y) \tag{4.3-3}$$

$$K' = +(k'_o)^2/(4\pi f'_Y) \tag{4.3-4}$$

$$I(y) = \int_{y_o}^{y} [n^2(\zeta) - 1]d\zeta \tag{4.3-5}$$

$$I(y') = \int_{y_o}^{y'} [n^2(\zeta) - 1]d\zeta \tag{4.3-6}$$

$$k_o = 2\pi f/c_o \tag{4.3-7}$$

$$k_o' = 2\pi f'/c_o \tag{4.3-8}$$

and $n^2(\cdot)$ is given by Eq. (4.1-42) where the expectation appearing in Eq. (4.3-2) is the characteristic function of the random quantity $[KI(y) + K'I(y')]$. If it is assumed that $I(y)$ is a real Gaussian random process, which implies that the index of refraction is a real Gaussian random process, then Eq. (4.3-2) can be written as

$$\begin{aligned} R_{H_M}(f,f',f_Y,f_Y';y,y') = {} & \left[2\pi(f_Y f_Y')^{1/2}\right]^{-1} \exp(+j[\Theta_{MD}(f,f_Y\ ;\ y) - \\ & \Theta_{MD}(f',f_Y'\ ;\ y')]) \cdot \\ & \exp(-E\{\Theta_{MR}^2(f,f_Y\ ;\ y)\}/2) \cdot \\ & \exp(+E\{\Theta_{MR}(f,f_Y;\ y)\Theta_{MR}(f',f_Y'\ ;\ y')\}) \cdot \\ & \exp(-E\{\Theta_{MR}^2(f',f_Y'\ ;\ y')\}/2) \end{aligned} \tag{4.3-9}$$

where

$$E\{\Theta_{MR}^2(f,f_Y\ ;\ y)\} = [k_o^2/(2\pi f_Y)]^2 \int_{y_o}^{y}\int_{y_o}^{y} n_D(\zeta)n_D(\zeta')\sigma(\zeta)\sigma(\zeta') \cdot R_{n_{NR}}(\zeta,\zeta')\,d\zeta d\zeta' \tag{4.3-10}$$

$$E\{\Theta_{MR}^2(f',f_Y' ; y')\} = [(k_0')^2/(2\pi f_Y')]^2 \int_{y_0}^{y'} \int_{y_0}^{y'} n_D(\zeta) n_D(\zeta')\sigma(\zeta)\sigma(\zeta') \cdot R_{n_{NR}}(\zeta,\zeta') d\zeta d\zeta' \tag{4.3-11}$$

$$E\{\Theta_{MR}(f,f_Y ; y)\Theta_{MR}(f',f_Y' ; y')\} = +[k_0^2/(2\pi f_Y)][(k_0')^2/(2\pi f_Y')] \cdot \int_{y_0}^{y} \int_{y_0}^{y'} n_D(\zeta) n_D(\zeta')\sigma(\zeta)\sigma(\zeta') \cdot R_{n_{NR}}(\zeta,\zeta') d\zeta d\zeta' \tag{4.3-12}$$

and

$$R_{n_{NR}}(y,y') = E\{n_{NR}(y) n_{NR}(y')\}. \tag{4.3-13}$$

If it is further assumed that the deterministic component $n_D(y)$ of the index of refraction is equal to unity [40, 42-45], and that the random component $n_R(y)$, and hence, the normalized random component $n_{NR}(y)$ is wide-sense stationary, i.e.,

$$R_{n_{NR}}(y,y') = R_{n_{NR}}(\Delta y) \tag{4.3-14}$$

where $\Delta y = y-y'$, then [see Eq. (4.1-45)]

$$\Theta_{MD}(f,f_Y ; y) = 0 \tag{4.3-15}$$

and

$$\Theta_{MD}(f',f_Y' \; ; \; y') = 0, \tag{4.3-16}$$

and Eqs. (4.3-10) thru (4.3-12) become, respectively,

$$E\left\{\Theta_{MR}^2(f,f_Y \; ; \; y)\right\} = (y-y_o)\left[\frac{k_o^2\sigma}{2\pi f_Y}\right]^2 \int_{-(y-y_o)}^{(y-y_o)} \left[1 - \frac{|\zeta|}{(y-y_o)}\right] R_{n_{NR}}(\zeta)d\zeta \tag{4.3-17}$$

$$E\left\{\Theta_{MR}^2(f',f_Y' \; ; \; y')\right\} = (y'-y_o)\left[\frac{(k_o')^2\sigma}{2\pi f_Y'}\right]^2 \int_{-(y'-y_o)}^{(y'-y_o)} \left[1 - \frac{|\zeta|}{(y'-y_o)}\right] R_{n_{NR}}(\zeta)d\zeta \tag{4.3-18}$$

and

$$E\left\{\Theta_{MR}(f,f_Y;y)\,\Theta_{MR}(f',f_Y' \; ; \; y')\right\} = \frac{(k_o k_o'\sigma)^2}{(2\pi f_Y)(2\pi f_Y')} \cdot$$

$$\left\{ (y-y')\int_{(y-y')}^{(y-y_o)} R_{n_{NR}}(\zeta)d\zeta + (y'-y_o)\int_{-(y'-y_o)}^{(y-y_o)} R_{n_{NR}}(\zeta)d\zeta + \right.$$

$$\left. \int_{-(y'-y_o)}^{0} \zeta R_{n_{NR}}(\zeta)d\zeta - \int_{(y-y')}^{(y-y_o)} \zeta R_{n_{NR}}(\zeta)d\zeta \right\} \tag{4.3-19}$$

where σ is the constant standard deviation of the Gaussian, zero mean, wide-sense stationary, random component $n_R(y)$ of the index of refraction.

V. SUMMARY and CONCLUSIONS

A consistent notation, fundamental input-output relations, and various time-space transformations for both deterministic and random linear, time-variant, space-variant filters have been established. The notation is consistent in the sense that all of the various input-output relations which are based upon the general theory will reduce to the classical relations of linear, time-invariant filter theory. These results should be of interest to persons involved in the general area of linear systems theory, and not only to those involved in underwater acoustics.

With the use of the method of separation of variables and the W.K.B. approximation, a mathematical expression of a time-invariant, space-variant, random transfer function of the ocean medium was derived. The transfer function was time-invariant instead of time-variant because motion was not considered in the present derivation. The transfer function modelled the ocean volume between transmit and receive apertures (arrays). The ocean volume was characterized by a random index of refraction (sound speed profile) which was a function of depth. The index of refraction was decomposed into deterministic and random components.

In addition to the transfer function derivation, two example calculations were made. The first example demonstrated the use of the coupling equations and involved the derivation of a mathematical expression for the random output electrical signal at each element in a receive planar array of complex weighted point sources. The output signals were expressed in terms of the frequency spectrum of the transmitted electrical signal, the transmit and receive arrays, and the previously derived transfer function of the ocean medium. The first example demonstrated that an output electrical signal could be derived in a logical and straightforward fashion. The second example involved the derivation of the coherence function, i.e., the autocorrelation function of the transfer function. In order to obtain somewhat simplified results, it was necessary to assume that the random component of the index of refraction was Gaussian and wide-sense stationary.

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